APPM 3570 / STAT 3100
Fall 2021
Exam 01
September 29

GENERAL INSTRUCTIONS:

- This exam has two parts, and you may start on either as long as you follow the instructions for each.
- Notes, your text and other books, cell phones, and other electronic devices are not permitted, except for calculators.
- Calculators are permitted.
- Write your name and sign and date your pledge to the CU Honor Code in the box below.

On my honor as a University of Colorado Boulder student, I have neither given nor received unauthorized assistance on either part of this work.
Name (Last, First): ______________________________
Signature: ________________________ **Date:** ____________

ADDITIONAL INSTRUCTIONS FOR PART I: Use the last few pages of your bluebook as scratch paper to work out the following 10 questions. For each question, circle what you believe to be the correct answer. Ambiguous answers to a question will result in 0 points for that question. Do not justify your answers to these 10 questions.

1. (4 points.) What's the coefficient of z^0 (i.e. the constant term) in the expansion of

$$\left(z + \frac{1}{z}\right)^{20} \quad ?$$

Solution: $\left(z+\frac{1}{z}\right)^{20} = \sum_{k=0}^{20} \binom{20}{k} z^k \left(\frac{1}{z}\right)^{20-k} = \sum_{k=0}^{20} \binom{20}{k} z^{2k-20}$. The coefficient of z^0 is associated with $k = 10$, which has the coefficient $\binom{20}{10}$.

2. (4 points.) Urn A has 4 green marbles and 3 red marbles. Urn B has 5 green marbles and 2 red marbles. A fair die is rolled. If the face shows a "2" then a marble is chosen at random from Urn A. Otherwise, a marble is chosen at random from Urn B. What is the probability that a green marble is selected?

Solution: Let T = the event that the die shows a 2. Let G be the event that a green ball is selected.

$$\begin{aligned} P(G) &= P(G|T)P(T) + P(G|T^C)P(T^C) \\ &= \left(\frac{4}{7}\right)\left(\frac{1}{6}\right) + \left(\frac{5}{7}\right)\left(\frac{5}{6}\right) \end{aligned}$$

$$= \frac{29}{42}.$$

3. (4 points.) The word "supercalifragilisticexpialidocious" means *extraordinarily good, or wonderful.* The letter count of this 34-letters word is given by the table below:

A	C	D	E	F	G	I	L	O	P	R	S	T	U	X
3	3	1	2	1	1	7	3	2	2	2	3	1	2	1

How many different words (in English or otherwise) may be constructed permuting the letters in "supercalifragilisticexpialidocious"?

Solution: Among the 34 possible positions, we need to select 3 to put A's, 3 to put C's, 1 to put a D, etc. The number of ways to do so is $\binom{34}{3,3,1,2,1,1,7,3,2,2,2,3,1,2,1}$.

4. (4 points) A scientific experiment is to be carried out into three consecutive phases, and the experiment as a whole is regarded as a success only if each phase is accomplished successfully. If

- the chance that phase I succeeds is 20%;
- the chance that phase II succeeds, given that phase I succeeded, is 30%; and
- the chance that phase III succeeds, given that phases I and II succeeded, is 40%;

what's the probability the experiment succeeds as a whole?

Solution: Let S_i denote the event that the "i-th phase was a success." Then the probability that the experiment succeeds as a whole is $P(S_1S_2S_3) = P(S_1) \cdot P(S_2|S_1) \cdot P(S_3|S_1S_2) = 0.2 \cdot 0.3 \cdot 0.4 = 0.024$.

5. (4 points.) After finishing his laundry, Martin had only 3 different pairs of socks left to pair when the lights went suddenly off. If he paired the 6 socks completely at random, what's the probability he paired all of them correctly?

Solution: We may represent outcomes of this experiment as partitions of $l_1, r_1, l_2, r_2, l_3, r_3$ into three sets of size-2. For instance, $\{\{r_3, l_2\}, \{r_1, l_1\}, \{l_3, r_2\}\}$ would indicate that only pair 1 was correctly paired. Then $|S| = \frac{1}{3!}\binom{6}{2,2,2}$, where the 3! is necessary because the order of the sets in the partition is irrelevant. With respect to this sample space, the event of interest contains a single outcome: $\{\{l_1, r_1\}, \{l_2, r_2\}, \{l_3, r_3\}\}$, and since all outcomes should be equally likely, its probability is: $\frac{1}{\frac{1}{3!}\binom{6}{2,2,2}}$.

6. (4 points.) A person has 7 friends of whom 4 will be invited to go sailing. How many choices are possible if 2 of the friends will only go sailing together?

Solution: $\binom{2}{2}\binom{5}{2} + \binom{2}{0}\binom{5}{4} = 10 + 5 = 15$.

7. (4 points.) Let A and B be two events associated with a random experiment, modeled with a probability measure P. If $P(A^c|B) = 0.2$ and $P(B) = 0.3$, what's $P(AB)$?

Solution: $P(A^cB) = P(A^c|B) \cdot P(B) = 0.2 \times 0.3 = 0.06$, and $P(AB) = P(B) - P(A^cB) = 0.3 - 0.06 = 0.24$.

8. (4 points.) 40% of students in a class wear a wristband of some type. 10% have a tattoo. 8% have both a wristband and a tattoo. What is the probability that a randomly selected person in the class has neither a wristband nor a tattoo?

Solution: $P(T^C W^C) = 1 - P(T \cup W) = 1 - (P(T) + P(W) - P(TW)) = 1 - (0.1 + 0.4 - 0.08) = 0.58$.

9. (4 points.) A discrete random variable X has probability mass function (p.m.f.):

$$p_X(x) = \begin{cases} c/3 & , \text{ if } x = \sqrt{2}; \\ 3c & , \text{ if } x = \sqrt{5}; \\ 0 & , \text{ otherwise}; \end{cases}$$

where c is certain constant. What value must c have?

Solution: Since X is discrete, we must have: $\frac{c}{3} + 3c = 1$, i.e. $\frac{10c}{3} = 1$, hence $c = 3/10$.

10. (4 points.) An urn has 4 purple balls, 8 green balls, 2 red balls and 1 white ball. A ball is randomly selected one at a time, without replacement, until the urn is empty. What is the probability that the last ball selected is purple?

Solution: The same as the probability that the first ball selected is purple, i.e. $\frac{4}{15}$.

*** TWO MORE QUESTIONS AHEAD! ***

ADDITIONAL INSTRUCTIONS FOR PART II: Use the <u>front</u> of your bluebook to answer to the following 2 questions. On the front cover of your bluebook:

- Write your name (Last, First).
- Write when your class meets (9 AM, or 3 PM).
- Draw a grading table with four rows and two columns.

In addition:

- Start each problem on a new page of your bluebook.
- Attempt all parts for each problem.
- [Box] in your answers.
- Show all your work and justify your answers.

Problem A. (30 points.) Three classmates, Jeanette, Sue, and Keisha, studied on their own until very late last night for an early exam. Assume that:

- there is a 4% chance that Jeanette oversleeps;
- a 2% chance that Sue oversleeps;
- a 1% chance that Keisha oversleeps; and
- each of them oversleeps (or not) independently of the others.

Respond to the following questions, fully simplifying your answers:

(a) What's the probability that Sue oversleeps given that Jeanette does? Explain!
(b) What's the probability that Jeanette oversleeps but Keisha doesn't? Explain!
(c) What's the probability that at least one of the three classmates oversleeps? Explain!

Solution:

(a) Define the events $J :=$"Jeanette oversleeps," $S :=$"Sue oversleeps," and $K :=$"Keisha oversleeps." Then $P(S|J) = P(S) = 2\%$ because S and J are independent.

(b) $P(JK^c) = P(J) \cdot P(K^c) = 4\% \times 99\% = 0.0396 = 3.96\%$, because J and K are independent, hence so are J and K^c.

(c)

$$\begin{aligned} P(J \cup S \cup K) &= 1 - P(J^c S^c K^c) \\ &= 1 - P(J^c) \cdot P(S^c) \cdot P(K^c) \\ &= 1 - 0.96 \cdot 0.98 \cdot 0.99 = 0.068608 \approx 6.8\% \end{aligned}$$

Problem B. (30 points.) Out of 100 coins, 99 are fair and one has heads on both sides. One coin is chosen at random and flipped twice.

(a) Given that heads is observed on both flips, what is the probability that the coin selected is a fair coin?

(b) Let X denote the number of heads observed in the experiment. Write the probability mass function (p.m.f.) of X.

(c) Write the cumulative distribution function (c.d.f.) of X.

Solution:

(a) $P(F|HH) = \frac{P(HH|F)P(F)}{P(HH|F)P(F)+P(HH|F^C)P(F^C)} = \frac{P(HHF)}{P(HHF)+P(HHF^C)} = \frac{(.5)(.5)(.99)}{(.5)(.5)(.99)+(1)(1)(.01)} = \frac{.2475}{.2575} \approx .9612$

(b)

pmf of X

x	p(x)
0	.2475
1	.4950
2	.2575

(c) The distribution function of X is given by

$$F(a) = \begin{cases} 0 & a < 0 \\ .2475 & 0 \leq a < 1 \\ .7425 & 1 \leq a < 2 \\ 1 & 2 \leq a \end{cases}$$