Write your name below. This exam is worth 100 points. On each problem, you must show all your work to receive credit on that problem. You are allowed to use one page of notes. You cannot collaborate on the exam or seek outside help, nor can you use the recorded lectures, a calculator, any computational software, or material you find online.

Name:

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1. (10 points, 2 each) If the statement is always true then write **TRUE**; if it is possible for the statement to be false then write **FALSE**. No justification is necessary.

   (a) The transformation $L : \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}$ defined by $L(A) = \dfrac{1}{2}(A + A^T)$ is a linear transformation.
   Solution: True. We find that $L(aA + bB) = (1/2)(aA + bB + aA^T + bB^T)$
   $= (1/2)(a(A + A^T) + b(B + B^T)) = aL(A) + bL(B)$

   (b) Every $n \times n$ rank 1 matrix can be written as uv^T for some $u, v \in \mathbb{R}^n$.
   Solution: True, since each row is a multiple of the first.

   (c) If A has Jordan Canonical Form J then A^2 has Jordan Canonical Form J^2.
   Solution: False Counterexample: $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} = J$,
   but $A^2 = \begin{pmatrix} 4 & 4 \\ 0 & 4 \end{pmatrix}$ has $J = \begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix} \neq \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}^2$

   (d) If $q(x) = x^T Kx - 2x^T f + c$ and K is positive semi-definite, then there are infinitely many vectors that minimize $q(x)$
   Solution: False. If $Kx = f$ has no solution, then there is no minimizer.

   (e) Let $K = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$ and $q(x) = x^T Kx$. The minimum value of $q(x)$ on the unit circle is -2.
   Solution: False. The minimum value is the minimal eigenvalue of K, which is $\dfrac{3 - \sqrt{17}}{2}$

2. (20 points) A linear transformation $L : \mathbb{R}^3 \to \mathbb{R}^2$ is represented in the standard basis by $A = \begin{pmatrix} 1 & -3 & 0 \\ -5 & -1 & -4 \end{pmatrix}$.

   (a) Find the matrices S and T such that $B = T^{-1}AS$ is in canonical form.

   (b) Let $b = \begin{pmatrix} 6 \\ -2 \\ 4 \end{pmatrix}$. Find the coordinates of $L(b)$ in the new basis.

Solution:

(a) For the S matrix we choose the first two columns of A^T to be a basis for coimg$A$ and the last column to be a basis for $\ker A$:

$\ker A$: $A \to \begin{pmatrix} 1 & -3 & 0 \\ 0 & -16 & -4 \end{pmatrix}$ so $s_3 = \begin{pmatrix} -3 \\ -1 \\ 4 \end{pmatrix}$.

$S = \begin{pmatrix} 1 & -5 & -3 \\ -3 & -1 & -1 \\ 0 & -4 & 4 \end{pmatrix}$

The columns of T are given by L acting on the columns of S:

$T = (As_1 As_2) = \begin{pmatrix} 10 & -2 \\ -2 & 42 \end{pmatrix}$

(b) We first find the coordinates of b in the S basis:

$Sx = b$: $\left(\begin{array}{ccc|c} 1 & -5 & -3 & 6 \\ -3 & -1 & -1 & 1 \\ 0 & -4 & 4 & 4 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -5 & -3 & 6 \\ 0 & -16 & -10 & 16 \\ 0 & 0 & 26/4 & 0 \end{array}\right)$ so $x = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

To find the coordinates of $L(b)$ in the T basis, we multiply:

$Ab = TBS^{-1}b = TBS^{-1}Sx = TBx$

Since $Bx = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are the coordinates of $L(b)$ in the T basis.

3. (25 points) Let $A = \begin{pmatrix} 3 & -4 & 1 & -1 \\ 0 & -1 & 0 & 0 \\ 3 & -5 & 3 & -2 \\ 2 & -5 & 2 & 0 \end{pmatrix}$. Can A be diagonalized? If it can, find the matrices Λ and S such that $A = S\Lambda S^{-1}$. If it cannot, find the Jordan Canonical form of A and the matrix S such that $A = SJS^{-1}$. You do not need to calculate S^{-1} in either case.

Solution:

We must determine whether A is complete. The characteristic equation for A is

$(\lambda + 1)(\lambda - 2)^3 = 0,$

so our eigenvalues are $\lambda = -1$ with algebraic multiplicity 1 and $\lambda = 2$ with algebraic multiplicity 3. Checking the geometric multiplicity for $\lambda = 2$:

$$A - 2I = \begin{pmatrix} 1 & -4 & 1 & -1 \\ 0 & -3 & 0 & 0 \\ 3 & -5 & 1 & -2 \\ 2 & -5 & 2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -4 & 1 & -1 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \text{ The only eigenvector is } v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}.$$

We only have one eigenvector for $\lambda = 2$, so A is an incomplete matrix and therefor cannot be diagonalized.

Since $\lambda = 2$ has an algebraic multiplicity of 3, our eigenvector will start a Jordan chain of length 3 and the Jordan Canonical Form will have a 3×3 Jordan block for this eigenvalue. We can immediately conclude that

$$J = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

To get S we need the eigenvector for $\lambda = -1$ and the rest of the Jordan Chain for $\lambda = 2$.

$\lambda = -1$:

$$A + I = \begin{pmatrix} 4 & -4 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 3 & -5 & 4 & -2 \\ 2 & -5 & 2 & 1 \end{pmatrix} \to \begin{pmatrix} 4 & -4 & 1 & -1 \\ 0 & -3 & 3/2 & 3/2 \\ 0 & 0 & 9/4 & -9/4 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \text{ So } v_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Generalized eigenvectors for $\lambda = 2$:

$(A - 2I)v_2 = v_1$:

$$\left(\begin{array}{cccc|c} 1 & -4 & 1 & -1 & 1 \\ 0 & -3 & 0 & 0 & 0 \\ 3 & -5 & 1 & -2 & -2 \\ 2 & -5 & 2 & -2 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & -4 & 1 & -1 & 1 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right). \text{ So } v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

$(A-2I)v_3 = v_2$:

$$\left(\begin{array}{cccc|c} 1 & -4 & 1 & -1 & 0 \\ 0 & -3 & 0 & 0 & 0 \\ 3 & -5 & 1 & -2 & 1 \\ 2 & -5 & 2 & -2 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & -4 & 1 & -1 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right). \text{ So } v_3 = \begin{pmatrix} 1/2 \\ 0 \\ -1/2 \\ 0 \end{pmatrix}.$$

So our S matrix is

$$S = \begin{pmatrix} 1 & 1 & 0 & 1/2 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & -1/2 \\ 1 & 2 & 0 & 0 \end{pmatrix}$$

4. (20 points) Find e^{At} where $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}$.

Solution: We must first diagonalize A. The characteristic equation is $\lambda(1-\lambda)(\lambda-2) = 0$, so we have $\lambda = 0, 1, 2$ and each eigenvalue is complete.

The eigenvectors are

$$\lambda = 0\colon\ v_0 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \qquad \lambda = 1\colon\ v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \lambda = 2\colon\ v_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

Our Λ and S matrices are

$$\Lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad S = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{pmatrix}$$

We calculate S^{-1} to be

$$S^{-1} = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 0 & -1/2 & 1/2 \end{pmatrix}$$

So we have $A = S\Lambda S^{-1}$ and therefore $e^{At} = Se^{\Lambda t}S^{-1}$:

$$e^{At} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^0 & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{2t} \end{pmatrix} \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 0 & -1/2 & 1/2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1/2 & 1/2 \\ e^t & 0 & 0 \\ 0 & -e^{2t}/2 & e^{2t}/2 \end{pmatrix}$$

$$= \begin{pmatrix} e^t & 0 & 0 \\ 0 & (1+e^{2t})/2 & (1-e^{2t})/2 \\ 0 & (1-e^{2t})/2 & (1+e^{2t})/2 \end{pmatrix}$$

5. (25 points) Let A be the matrix with the Singular Value Decomposition given by

$$A = \begin{pmatrix} 2/\sqrt{10} & 2/\sqrt{10} \\ -1/\sqrt{10} & 1/\sqrt{10} \\ -2/\sqrt{10} & 2/\sqrt{10} \\ 1/\sqrt{10} & 1/\sqrt{10} \end{pmatrix} \begin{pmatrix} 4\sqrt{5} & 0 \\ 0 & 2\sqrt{5} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 0 & 0 & -1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{pmatrix}$$

(a) (3 points) What is the rank of A?
Solution: A has 2 singular values, so $\operatorname{rank} A = 2$.

(b) (3 points) Is A singular or nonsingular?
Solution: A is 4×4 but $\operatorname{rank} A = 2 < 4$, so A is singular.

(c) (9 points) Find the best rank 1 approximation of A.
Solution: Using the first (largest) sindular value and its associated vectors gives:

$$\tilde{A} = \begin{pmatrix} 2/\sqrt{10} \\ -1/\sqrt{10} \\ -2/\sqrt{10} \\ 1/\sqrt{10} \end{pmatrix} 4\sqrt{5} \begin{pmatrix} 1/\sqrt{2} & 0 & 0 & -1/\sqrt{2} \end{pmatrix} = \frac{4\sqrt{5}}{\sqrt{10}\sqrt{2}} \begin{pmatrix} 2 & 0 & 0 & -2 \\ -1 & 0 & 0 & 1 \\ -2 & 0 & 0 & 2 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 0 & 0 & -4 \\ -2 & 0 & 0 & 2 \\ -4 & 0 & 0 & 4 \\ 2 & 0 & 0 & -2 \end{pmatrix}$$

(d) (10 points) Find the pseudoinverse of A and use it to calculate the least squares solution to $Ax = b$ where $b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

Solution: $A^+ = \begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} \\ -1/\sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} 1/4\sqrt{5} & 0 \\ 0 & 1/2\sqrt{5} \end{pmatrix} \begin{pmatrix} 2/\sqrt{10} & -1/\sqrt{10} & -2/\sqrt{10} & 1/\sqrt{10} \\ 2/\sqrt{10} & 1/\sqrt{10} & 2/\sqrt{10} & 1/\sqrt{10} \end{pmatrix}$

$$= \frac{1}{\sqrt{2}(4\sqrt{5})\sqrt{10}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & -1 & -2 & 1 \\ 2 & 1 & 2 & 1 \end{pmatrix}$$

$$= \frac{1}{40} \begin{pmatrix} 2 & -1 & -2 & 1 \\ -4 & -2 & -4 & -2 \\ 4 & 2 & 4 & 2 \\ -2 & 1 & 2 & -1 \end{pmatrix}$$

The least squares solution to $Ax = b$ is

$$A^+ b = \frac{1}{10} \begin{pmatrix} 0 \\ -3 \\ 3 \\ 0 \end{pmatrix}$$