Write your name below. This exam is worth 100 points. On each problem, you must show all your work to receive credit on that problem. You are allowed to use one page of notes. You cannot collaborate on the exam or seek outside help, nor can you use the recorded lectures, a calculator, any computational software, or material you find online.

Name:

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1. (10 points, 2 each) If the statement is always true then write **TRUE**; if it is possible for the statement to be false then write **FALSE**. No justification is necessary.

   (a) For $n \times n$ matrix $A$, $||A^2||_\infty = ||A||_\infty^2$.
   **Solution: False**: A counterexample is $A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$, so we have $A^2 = \begin{pmatrix} 5 & 5 \\ 5 & 10 \end{pmatrix}$. $\|A\|_\infty = 1 + 3 = 4$, but $\|A^2\| = 5 + 10 = 15 < 4^2$.

   (b) If regular symmetric matrix $K$ has no null directions, then $K$ is either positive definite or negative definite.
   **Solution: True**: All of $K$'s pivots must be the same sign for it to not have null directions. If they are all positive, $K > 0$ and if they are all negative $K < 0$.

   (c) For all vectors $v$ and $w$ in a normed space, $\|v - w\| \le |\|v\| - \|w\||$.
   **Solution: False**: Counterexample: Let $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $w = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, then $\|v - w\| = \sqrt{2}$ but $|\|v\| - \|w\|| = |1 - 1| = 0$

   (d) If $A$ has a $QR$ factorization, then $R^T R$ is the Cholesky factorization of $A^T A$.
   **Solution: True**: $A^T A = (QR)^T (QR) = R^T Q^T Q R = R^T R$ Since $R$ is upper triangular, this is the Cholesky factorization.

   (e) The matrix $\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$ defines an inner product on $\mathbb{R}^2$.
   **Solution: True**: Factoring $K = \begin{pmatrix} 1 & 0 \\ 1/3 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 5/3 \end{pmatrix} \begin{pmatrix} 1 & 1/3 \\ 0 & 1 \end{pmatrix}$. As all pivots are positive, $K > 0$ and so defines an inner product.

2. (25 points) Consider the set of three vectors: $\left\{ \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} \right\}$

   (a) (9 points) Are these vectors linearly independent? Justify your answer.
   **Solution:** No, they are linearly dependent.

$$\begin{pmatrix} 2 & 1 & 1 \\ -1 & -2 & 4 \\ 3 & 1 & 3 \end{pmatrix} \to \begin{pmatrix} 2 & 1 & 1 \\ 0 & -3/2 & 9/2 \\ 0 & 0 & 0 \end{pmatrix}$$

   (b) (8 points) Show that $\langle x, y \rangle = x^T K y$ is an inner product on $\mathbb{R}^3$ when $K = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$.

   **Solution:** K is symmetric and factors into

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

   Since all of the pivots are positive, $K > 0$.

   (c) (8 points) Verify the triangle inequality holds for the first two vectors in our set using the inner product given in (b).
   **Solution:** After finding $x^T K x$ for each of our vectors and taking the square root we have:

$$\| \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \| = \sqrt{34} \qquad \| \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \| = \sqrt{19}$$

$$\| \begin{pmatrix} 2+1 \\ -1-2 \\ 3+1 \end{pmatrix} \| = \| \begin{pmatrix} 3 \\ -3 \\ 4 \end{pmatrix} \| = \sqrt{101}$$

   So the triangle inequality states that $\sqrt{101} \leq \sqrt{19} + \sqrt{34}$, which is true:

$$101 \leq 19 + 2\sqrt{19}\sqrt{34} + 34$$

$$48 \leq 2\sqrt{19}\sqrt{34}$$

$$24 \leq \sqrt{19}\sqrt{34}$$

$$576 \leq 646$$

3. (15 points) Let $A = \begin{pmatrix} 1 & 9 & 12 & 7 \\ 2 & 11 & -18 & -7 \\ 0 & 2 & 12 & 6 \end{pmatrix}$

   (a) (5 points) Find a basis for the image of $A$
   **Solution:** We need to find the REF of $A$:

   $$A = \begin{pmatrix} 1 & 9 & 12 & 7 \\ 2 & 11 & -18 & -7 \\ 0 & 2 & 12 & 6 \end{pmatrix} \to \begin{pmatrix} 1 & 9 & 12 & 7 \\ 0 & -1 & -6 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix} = U$$

   So $A$ has the first two columns as pivot columns, so the image has these columns as basis vectors.

   $$\text{img}A = \text{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 9 \\ 11 \\ 2 \end{pmatrix} \right\}$$

   (b) (5 points) Find a basis for the coimage of $A$
   **Solution:** The basis vectors of the coimage are the transposes of the nonzero rows of $U$:

   $$\text{coimg}A = \text{Span}\left\{ \begin{pmatrix} 1 \\ 9 \\ 12 \\ 7 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ -6 \\ -3 \end{pmatrix} \right\}$$

   (c) (3 points) What is the dimension of the cokernel of $A$
   **Solution:** $A$ is $3 \times 4$ and $\text{rank}A = 2$ so the cokernel has dimension $3 - 2 = 1$.

   (d) (2 points) What is the rank of $A^T$?
   **Solution:** $\text{rank}A^T$ is the same as $\text{rank}A = 2$.

4. (30 points) Let $A = \begin{pmatrix} 2 & 4 & -2 \\ 2 & 0 & -4 \\ -1 & -1 & 6 \end{pmatrix}$ and $b \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

   (a) (15 points) Find the $QR$ factorization of $A$.

   (b) (10 points) Use your factorization to solve $Ax = b$

   (c) (5 points) After calculating the first Householder matrix that factors $A$ into $QR$, you find that

   $$H_1 A = \begin{pmatrix} 3 & 3 & -6 \\ 0 & 2 & 4 \\ 0 & -2 & 2 \end{pmatrix}.$$

   Find the unit vector needed to make the second Householder matrix which factors $A$. You do not need to calculate the matrix.

**Solution:**

(a) Following our procedure to calculate the entries in $R$ and the columns of $Q$ gives:

$$r_{11} = \|a_1\| = \sqrt{4+4+1} = 3$$

$$q_1 = \frac{a_1}{r11} = \frac{1}{3}\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$$

$$r_{12} = \langle q_1, a_2 \rangle = \frac{1}{3}(8+0+1) = 3$$

$$r_{22} = \sqrt{\|a_2\|^2 - r_{12}^2} = \sqrt{16+0+1-9} = 2\sqrt{2}$$

$$q_2 = \frac{a_2 - r_{12}q_1}{r_{22}} = \frac{1}{2\sqrt{2}}\left(\begin{pmatrix} 4 \\ 0 \\ -1 \end{pmatrix} - 3\frac{1}{3}\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$r_{13} = \langle q_1, a_3 \rangle = \frac{1}{3}(-4-8-6) = -6$$

$$r_{23} = \langle q_2, a_3 \rangle = \frac{1}{\sqrt{2}}(-2+4+0) = \frac{2}{\sqrt{2}} = \sqrt{2}$$

$$r_{33} = \sqrt{\|a_3\|^2 - r_{13}^2 - r_{23}^2} = \sqrt{4+16+36-36-2} = \sqrt{18} = 3\sqrt{2}$$

$$q_3 = \frac{a_3 - r_{13}q_1 - r_{23}q_2}{r33} = \frac{1}{3\sqrt{2}}\left(\begin{pmatrix} -2 \\ -4 \\ 6 \end{pmatrix} - (-6)\frac{1}{3}\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} - \sqrt{2}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}\right)$$

$$q_3 = \frac{1}{3\sqrt{2}}\left(\begin{pmatrix} -2 \\ -4 \\ 6 \end{pmatrix} + \begin{pmatrix} 4 \\ 4 \\ -2 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}\right) = \frac{1}{3\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$$

Putting this all together gives:

$$A = \begin{pmatrix} 2/3 & 1/\sqrt{2} & 1/3\sqrt{2} \\ 2/3 & -1/\sqrt{2} & 1/3\sqrt{2} \\ -1/3 & 0 & 4/3\sqrt{2} \end{pmatrix}\begin{pmatrix} 3 & 3 & -6 \\ 0 & 2\sqrt{2} & \sqrt{2} \\ 0 & 0 & 3\sqrt{2} \end{pmatrix}$$

(b) Here we need to solve $Rx = Q^T b$ :

$$Q^T b = \begin{pmatrix} 2/3 & 2/3 & -1/3 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/3\sqrt{2} & 1/3\sqrt{2} & 4/3\sqrt{2} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \sqrt{2} \end{pmatrix}$$

We form the augmented matrix for $Rx = Q^T b$ and solve:

$$\left(\begin{array}{ccc|c} 3 & 3 & -6 & 1 \\ 0 & 2\sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & 3\sqrt{2} & \sqrt{2} \end{array}\right) \to \left(\begin{array}{ccc|c} 3 & 3 & -6 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 3 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 3 & 3 & 0 & 3 \\ 0 & 2 & 0 & -1/3 \\ 0 & 0 & 3 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 3 & 0 & 0 & 7/2 \\ 0 & 2 & 0 & -1/3 \\ 0 & 0 & 3 & 1 \end{array}\right)$$

So we have

$$x = \begin{pmatrix} 7/6 \\ -1/6 \\ 1/3 \end{pmatrix}$$

(c) We take the second column of the $H_1 A$ matrix and set the first entry equal to 0:
$\tilde{a_2} = \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix}$, which has a norm of $2\sqrt{2}$.

This vector needs to be reflected into the $e_2$ direction, so our unit vector is given by $u = \dfrac{\tilde{a_2} - 2\sqrt{2}e_2}{\|\tilde{a_2} - 2\sqrt{2}e_2\|}$

$$\tilde{a_2} - 2\sqrt{2}e_2 = \begin{pmatrix} 0 \\ 2 - 2\sqrt{2} \\ 2 \end{pmatrix}$$

$$\|\tilde{a_2} - 2\sqrt{2}e_2\| = 2\sqrt{(1-\sqrt{2})^2 + 1} = 2\sqrt{1 - 2\sqrt{2} + 2 + 1} = 2\sqrt{4 - 2\sqrt{2}}$$

So our unit vector is: $u = \dfrac{1}{\sqrt{4 - 2\sqrt{2}}} \begin{pmatrix} 0 \\ 1 - \sqrt{2} \\ 1 \end{pmatrix}$

5. (20 points) Let $A = \begin{pmatrix} 2 & 3 & -2 \\ 0 & -1 & 2 \\ 0 & -3 & 4 \end{pmatrix}$

   (a) (6 points) Find all the eigenvalues of $A$

   (b) (12 points) Find all the eigenvectors of $A$

   (c) (2 points) Is $A$ complete?

**Solution:**

(a) The characteristic polynomial of $A$ is

$$p(\lambda) = \det(A - \lambda I) = (2-\lambda)\det\begin{pmatrix} -1-\lambda & 2 \\ -3 & 4-\lambda \end{pmatrix}$$

$$= (2-\lambda)\left((-1-\lambda)(4-\lambda) - 2(-3)\right)$$

$$= (2-\lambda)(-4+\lambda-4\lambda+\lambda^2+6)$$

$$= (2-\lambda)(\lambda^2-3\lambda+2)$$

$$= (2-\lambda)(\lambda-2)(\lambda-1)$$

So the eigenvalues are $\lambda = 2$ with algebraic multiplicity of 2 and $\lambda = 1$ with algebraic multiplicity of 1.

(b) Eigenvectors:

$\lambda = 2$:
$A - 2I = \begin{pmatrix} 0 & 3 & -2 \\ 0 & -3 & 2 \\ 0 & -3 & 2 \end{pmatrix} \to \begin{pmatrix} 0 & 3 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, so we have two free variables and the eigenvectors are

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ and } v_2 = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$$

$\lambda = 1$:
$A - I = \begin{pmatrix} 1 & 3 & -2 \\ 0 & -2 & 2 \\ 0 & -3 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & -2 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{pmatrix}$, so our single eigenvector is

$$v_3 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

(c) Yes, all eigenvectors have the same algebraic multiplicities as geometric multiplicities, so they are all complete, so the matrix is complete.