Write your name below. This exam is worth 100 points. On each problem, you must show all your work to receive credit on that problem. You are allowed to use one page of notes. You cannot collaborate on the exam or seek outside help, nor can you use the recorded lectures, a calculator, any computational software, or material you find online.

Name:

1. (10 points, 2 each) If the statement is always true then write **TRUE**; if it is possible for the statement to be false then write **FALSE**. No justification is necessary.

   (a) For $n \times n$ matrix A, $||A^2||_\infty = ||A||_\infty^2$.

   (b) If symmetric matrix K has no null directions, then K is either positive definite or negative definite.

   (c) For all vectors v and w in a normed space, $\|v - w\| \leq |\|v\| - \|w\||$.

   (d) If A has a QR factorization, then $R^T R$ is the Cholesky factorization of $A^T A$.

   (e) The matrix $\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$ defines an inner product on $\mathbb{R}^2$.

2. (25 points) Consider the set of three vectors: $\left\{ \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} \right\}$

   (a) (9 points) Are these vectors linearly independent? Justify your answer.

   (b) (8 points) Show that $\langle x, y \rangle = x^T K y$ is an inner product on $\mathbb{R}^3$ when $K = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$.

   (c) (8 points) Verify the triangle inequality holds for the first two vectors in our set using the inner product given in (b).

3. (15 points) Let $A = \begin{pmatrix} 1 & 9 & 12 & 7 \\ 2 & 11 & -18 & -7 \\ 0 & 2 & 12 & 6 \end{pmatrix}$

   (a) (5 points) Find a basis for the image of A

   (b) (5 points) Find a basis for the coimage of A

   (c) (3 points) What is the dimension of the cokernel of A

   (d) (2 points) What is the rank of A^T?

4. (30 points) Let $A = \begin{pmatrix} 2 & 4 & -2 \\ 2 & 0 & -4 \\ -1 & -1 & 6 \end{pmatrix}$ and $b \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

   (a) (15 points) Find the QR factorization of A.

   (b) (10 points) Use your factorization to solve $Ax = b$

   (c) (5 points) After calculating the first Householder matrix that factors A into QR, you find that

   $$H_1 A = \begin{pmatrix} 3 & 3 & -6 \\ 0 & 2 & 4 \\ 0 & -2 & 2 \end{pmatrix}.$$

   Find the unit vector needed to make the second Householder matrix which factors A. You do not need to calculate the matrix.

5. (20 points) Let $A = \begin{pmatrix} 2 & 3 & -2 \\ 0 & -1 & 2 \\ 0 & -3 & 4 \end{pmatrix}$

   (a) (6 points) Find all the eigenvalues of A

   (b) (12 points) Find all the eigenvectors of A

   (c) (2 points) Is A complete?