APPM 3310, Exam 1, Summer 2024 Exam 1 Solutions

Write your name below. This exam is worth 100 points. On each problem, you must show all your work to receive credit on that problem. You are allowed to use one page of notes. You cannot collaborate on the exam or seek outside help, nor can you use the recorded lectures, a calculator, any computational software, or material you find online.

Name:

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1. (10 points, 2 each) If the statement is always true then write **TRUE**; if it is possible for the statement to be false then write **FALSE**. No justification is necessary.

   (a) A square matrix always commutes with its transpose.

   Solution: False. A counterexample is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$:
   $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ but $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$

   (b) If A is invertible and $c \in \mathbb{R}$ is non-zero, then $(cA)^{-1} = \frac{1}{c}A^{-1}$

   Solution: True. $\frac{1}{c}A^{-1}(cA) = \frac{1}{c}cA^{-1}A = I$ and $cA\frac{1}{c}A^{-1} = \frac{1}{c}cAA^{-1} = I$

   (c) If A and B are n $\times$ n symmetric matrices, then AB is also symmetric.

   Solution: False. $(AB)^T = B^T A^T = BA$. This only equals AB if A and B commute.

   (d) If A is skew-symmetric, then $\det(A) = 0$.

   Solution: False. The skew-symmetric matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ has a determinate of 1.

   (e) For matrices A and B, $AB = O$ implies either $A = O$ or $B = O$.

   Solution: False. A counterexample is $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

2. (25 points) Let $A = \begin{pmatrix} 2 & -6 & 4 \\ -1 & 3 & 1 \\ 1 & -5 & 6 \end{pmatrix}$ and $b = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}$

   (a) (15 points) If A is regular find its LU factorization. If A is not regular but is non-singular, find a permuted LU factorization.

   (b) (5 points) Find the determinant of A using the factorization from (a).

   (c) (5 points) Solve the equation $Ax = b$.

Solutions:

(a) $A = \begin{pmatrix} 2 & -6 & 4 \\ -1 & 3 & 1 \\ 1 & -5 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 1/2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & -6 & 4 \\ 0 & 0 & 3 \\ 0 & -2 & 4 \end{pmatrix}$

We see that A is not regular, so we must permute the second and third rows. Our permuted factorization is now

$$PA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & -6 & 4 \\ -1 & 3 & 1 \\ 1 & -5 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ -1/2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & -6 & 4 \\ 0 & -2 & 4 \\ 0 & 0 & 3 \end{pmatrix} = LU$$

(b) We had to do one row interchange, so the determinant is (-1) times the product of A's pivots:

$\det(A) = (-1)(2)(-2)(3) = 12$

(c) We must first apply the permutation to our b vector:

$$Pb = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} = LUx$$

Then we solve $Lc = Pb$:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 2 \\ -1/2 & 0 & 1 & 3 \end{array}\right) \xrightarrow[R_3=R_3+\frac{1}{2}R_1]{R_2=R_2-\frac{1}{2}R_1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right) \text{ so } c = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$$

Finally we solve $Ux = c$:

$$\left(\begin{array}{ccc|c} 2 & -6 & 4 & 0 \\ 0 & -2 & 4 & 2 \\ 0 & 0 & 3 & 3 \end{array}\right) \xrightarrow[R_2=R_2-\frac{4}{3}R_3]{R_1=R_1-\frac{4}{3}R_3} \left(\begin{array}{ccc|c} 2 & -6 & 0 & -4 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & 3 & 3 \end{array}\right) \xrightarrow{R_1=R_1-3R_2} \left(\begin{array}{ccc|c} 2 & 0 & 0 & 2 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & 3 & 3 \end{array}\right)$$

so $x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

The system could also be solved directly.

3. (20 points) The following two questions are unrelated.

   (a) (10 points) Are the polynomials $\{x^2+1, x^2-1, x\}$ a basis for $\mathcal{P}^2$, the vector space of all polynomials with degree ≤ 2? Justify your answer.

   (b) (10 points) Find a basis for the span of $\left\{\begin{pmatrix}1\\1\\2\\-1\end{pmatrix}, \begin{pmatrix}3\\2\\7\\0\end{pmatrix}, \begin{pmatrix}2\\1\\5\\1\end{pmatrix}\right\}$.

   Are these vectors linearly independent? Justify your answer.

Solution:

(a) We determine if a general quadratic can be written as a linear combination of the polynomials:

$ax^2 + bx + d = c_1(x^2+1) + c_2(x^2-1) + c_3x$
$= (c_1+c_2)x^2 + c_3x + (c_1-c_2)$
equating terms of the same degree produces the linear system:
$c_1 + c_2 = a$
$c_3 = b$
$c_1 - c_2 = d$
Our coeficient matrix is therefore $A = \begin{pmatrix}1 & 1 & 0\\0 & 0 & 1\\1 & -1 & 0\end{pmatrix}$, which is non-singular.
$\begin{pmatrix}1 & 1 & 0\\0 & 0 & 1\\1 & -1 & 0\end{pmatrix} \to \begin{pmatrix}1 & 1 & 0\\0 & 0 & 1\\0 & -2 & 0\end{pmatrix} \to \begin{pmatrix}1 & 1 & 0\\0 & -2 & 0\\0 & 0 & 1\end{pmatrix}$.
So for any quadratic with coefficients a, b, and d, we can calculate the coefficients to write it as a linear combination of our vectors, so they do span $\mathcal{P}^2$.

(b) We create the matrix A whose columns are made up of these vectors:

$$A = \begin{pmatrix}1 & 3 & 2\\1 & 2 & 1\\2 & 7 & 5\\-1 & 0 & 1\end{pmatrix}$$

and find the row-echelon form:

$$\begin{pmatrix}1 & 3 & 2\\1 & 2 & 1\\2 & 7 & 5\\-1 & 0 & 1\end{pmatrix} \to \begin{pmatrix}1 & 3 & 2\\0 & -1 & -1\\0 & 1 & 1\\0 & 3 & 3\end{pmatrix} \to \begin{pmatrix}1 & 3 & 2\\0 & -1 & -1\\0 & 0 & 0\\0 & 0 & 0\end{pmatrix}.$$

From this, we conclude that we can express the third vector as a linear combination of the first two, so the vectors are linearly dependent. The first two columns contain the pivots, so a basis for the span of these vectors is the set containing first two: $\left\{\begin{pmatrix}1\\1\\2\\-1\end{pmatrix}, \begin{pmatrix}3\\2\\7\\0\end{pmatrix}\right\}$

4. (20 points) Consider the vector spaces C^1, consisting of all continuously differentiable scalar functions f, and $\mathcal{M}_{3\times 3}$, the set of all 3×3 matrices. Determine if the sets of functions or matrices which satisfy the following conditions are subspaces.

   (a) (5 points) $f'(x) = f(x) + x$

   (b) (5 points) $f(-x) = e^{2x} f(x)$

   (c) (5 points) 3×3 matrices of the form $\begin{pmatrix} a & 0 & b \\ 0 & ab & 0 \\ b & 0 & a \end{pmatrix}$ where $a, b \in \mathbb{R}$.

   (d) (5 points) Singular 3×3 matrices.

   Solutions:

   (a) This is not a subspace as it does not contain the 0 function and fails both closure tests. For example:
   Let $g(x) = cf(x)$ then $g'(x) = cf'(x) = c(f(x) + x) = cf(x) + cx = g(x) + cx \neq g(x) + x$
   So this set is not closed under scalar multiplication, and therefor is not a subspace.

   (b) This is a subspace. The constant function $f(x) = 0$ is the 0 vector and is in our set. It is also closed under both scalar multiplication and vector addition:

   Let $g(x) = cf(x)$ then $g(-x) = cf(-x) = ce^{2x} f(x) = e^{2x} cf(x) = e^{2x} g(x)$.

   Let $h(x) = f(x) + g(x)$ with both f and g in our set, then
   $h(-x) = f(-x) + g(-x) = e^{2x} f(x) + e^{2x} g(x) = e^{2x}(f(x) + g(x)) = e^{2x} h(x)$

   (c) This is not a subspace. It contains the 0 vector, but is not closed under either scalar multiplication or vector addition. For example:

   $$c \begin{pmatrix} a & 0 & b \\ 0 & ab & 0 \\ b & 0 & a \end{pmatrix} = \begin{pmatrix} ca & 0 & cb \\ 0 & cab & 0 \\ cb & 0 & ca \end{pmatrix} \neq \begin{pmatrix} ca & 0 & cb \\ 0 & (ca)(cb) & 0 \\ cb & 0 & ca \end{pmatrix}$$

   (d) This is not a subspace, as there are many singular matrices that can be added together to get a non-singular matrix. For example:

   $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3$$

5. (25 points) Let $A = \begin{pmatrix} 1 & -2 & -6 & 9 \\ 1 & 2 & 2 & -1 \\ -1 & 0 & 2 & -4 \\ -1 & 2 & 6 & -9 \end{pmatrix}$

   (a) (7 points) Find a basis for the image of A

   (b) (7 points) Find a basis for the kernel of A

   (c) (6 points) Find a basis for the coimage of A

   (d) (5 points) What is the rank of A^T?

Solutions:
We must find the REF of A to answer these questions:

$$\begin{pmatrix} 1 & -2 & -6 & 9 \\ 1 & 2 & 2 & -1 \\ -1 & 0 & 2 & -4 \\ -1 & 2 & 6 & -9 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & -6 & 9 \\ 0 & 4 & 8 & -10 \\ 0 & -2 & -4 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & -6 & 9 \\ 0 & 4 & 8 & -10 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(a) Image: The first two columns are our pivot columns, so they are our basis:

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \\ 0 \\ 2 \end{pmatrix} \right\}$$

(b) Kernel: We have two free variables, z_3 and z_4, and hence two basis vectors. We solve the homogeneous equation twice, setting one of the free variables equal to zero each time:

$$\left\{ \begin{pmatrix} 2 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 5/2 \\ 0 \\ 1 \end{pmatrix} \right\}$$

(c) Coimage: The basis vectors for the cokernel are the transposes of the non-zero rows of the REF:

$$\left\{ \begin{pmatrix} 1 \\ -2 \\ -6 \\ 9 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ 8 \\ -10 \end{pmatrix} \right\}$$

(d) The rank of A^T is the same as rank A, which is 2