Write your name below. This exam is worth 100 points. On each problem, you must show all your work to receive credit on that problem. You may use your notes, the course textbook, and the recorded lectures for this exam, but no other resources. You are not allowed to collaborate on the exam or seek outside help, nor can you use any other notes, other books, a calculator, any computational software, or any other materials you find online. Be sure to submit your work to Gradescope by 11:59pm (Mountain Time) on Friday July 22.

Name:

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1. (32 points: 8 each) If the statement is **always true** mark "TRUE" and provide a *brief* justification; if it is possible for the statement to be false then mark "FALSE" and provide a **counterexample.**

   (a) Suppose that A is a square matrix with real entries, real eigenvalues, and an orthogonal eigenbasis. True or False: A is symmetric.
   Solution:

   True. Let $\{\mathbf{u}_1, ..., \mathbf{u}_n\}$ be the orthogonal eigenbasis of A with corresponding eigenvalues $\lambda_1, ..., \lambda_n$. Let $Q = [\mathbf{u}_1 \cdots \mathbf{u}_n]$ and Λ be the diagonal matrix with $\lambda_1, ..., \lambda_n$ as the diagonal entries. Then, $Q\Lambda Q^T$ is the spectral decomposition of $A = Q\Lambda Q^T$, and A must be symmetric.

   (b) Suppose that Q is a real $n \times n$ orthogonal matrix and $A = QB$ where A and B are both real $n \times p$ matrices. True or false: A and B have the same singular values.
   Solution:

   True.

$$\begin{aligned} A^TA &= (QB)^TQB \\ &= B^TQ^TQB \\ &= B^TB. \end{aligned}$$

   Since $A^TA = B^TB$, then A^TA and B^TB have the same eigenvalues, then A and B must have the same singular values.

   (c) Consider $A\mathbf{x} = \mathbf{b}$. The least squares solution of $A\mathbf{x} = \mathbf{b}$ is unique.
   Solution:

   False. As a counterexample, consider the linear system

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

If we write $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, then $A^T A\mathbf{x} = A^T\mathbf{b}$ has solutions

$$\mathbf{x} = \begin{bmatrix} \frac{3}{2} \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

(d) Consider the quadratic equation given by

$$p(\mathbf{x}) = \mathbf{x}^T \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \mathbf{x} - 2\mathbf{x}^T \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3.$$

True or false: It has an absolute minimum value of 2.

Solution:

True. We can show that $K = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$ is positive definite by noting that it is symmetric, and then showing that it is regular and has positive pivots. So, $K\mathbf{x} = \mathbf{f}$ will have a unique solution of $\mathbf{x}^* = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. This is the unique minimizer of $p(\mathbf{x})$. So, the absolute minimum value of $p(\mathbf{x})$ is

$$p(\mathbf{x}^*) = 3 - (\mathbf{x}^*)^T \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \mathbf{x}^* = 2.$$

2. (24 points) Consider the linear system $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{bmatrix} 2 \\ 5 \\ 6 \\ 6 \end{bmatrix}.$$

(a) Is $A\mathbf{x} = \mathbf{b}$ consistent?
(b) Does $\mathbf{b}$ lie in the image of A?
(c) Find the least squares solution of $A\mathbf{x} = \mathbf{b}$.
(d) What is the orthogonal projection of $\mathbf{b}$ onto the image of A?
(e) What is the closest point to $\mathbf{b}$ in the image of A?
(f) Including multiplicities, what are the singular values of A?

Solution:

(a) We will consider the augmented matrix $[A|\mathbf{b}]$ and reduce it until we can see if the system is consistent or not.

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 1 & 0 & -1 & 5 \\ 0 & 1 & 1 & 6 \\ -1 & 1 & -1 & 6 \end{array}\right] \xrightarrow[R_4'=R_4+R_1]{R_2'=R_2-R_1} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & -1 & -1 & 3 \\ 0 & 1 & 1 & 6 \\ 0 & 2 & -1 & 8 \end{array}\right] \xrightarrow{R_3'=R_3+R_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & -1 & -1 & 3 \\ 0 & 0 & 0 & 9 \\ 0 & 2 & -1 & 8 \end{array}\right]$$

Note that the third row implies $0 = 9$ which is clearly a contradiction. So, this linear system is not consistent.

(b) No. $\mathbf{b}$ cannot lie in the image of A because there is not $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{b}$.

(c) We must solve $A^T A\mathbf{x} = A^T\mathbf{b}$.
We see that

$$A^T A = 3I_3 \qquad \text{and} \qquad A^T\mathbf{b} = \begin{bmatrix} 1 \\ 14 \\ -5 \end{bmatrix}.$$

Solving $A^T A\mathbf{x} = A^T\mathbf{b}$, we find $\mathbf{x} = \frac{1}{3}\begin{bmatrix} 1 \\ 14 \\ -5 \end{bmatrix}$.

(d) Note that the columns of A are linearly independent. Using the value $\mathbf{x}$ that we have just obtained, the orthogonal projection of $\mathbf{b}$ onto the image of A is

$$\begin{aligned} \mathbf{w} &= A(A^T A)^{-1}A^T\mathbf{b} \\ &= A\mathbf{x} \\ &= \begin{bmatrix} 5 \\ 2 \\ 3 \\ 6 \end{bmatrix}. \end{aligned}$$

(e) The closest point is also the orthgonal projection, so it must be $\begin{bmatrix} 5 \\ 2 \\ 3 \\ 6 \end{bmatrix}$.

(f) Since $A^T A = 3I_3$, we see the eigenvalues of $A^T A$ (including multiplicities) are 3, 3, and 3. So, the singular values of A are $\sqrt{3}$, $\sqrt{3}$, and $\sqrt{3}$.

3. (22 points) Consider

$$A = \begin{bmatrix} -2 & 0 & 0 \\ -3 & -2 & -3 \\ 3 & 0 & 1 \end{bmatrix}.$$

(a) Is A complete?

(b) Use a factorization of A to help determine A^{112}. (Your final answer should be written as a 3×3 matrix, but the entries of the matrix do not need to simplified. Reminder: you may not use a calculator or computational software on this exam.)

(c) Find e^{tA}

Solution:

(a) We find $\det(A - \lambda I) = (-2-\lambda)^2(1-\lambda)$, so the eigenivalues of A are -2 and 1. The algebraic multiplicities are 2 and 1, respectively. So, for A to be complete, we need only determine if $\lambda = -2$ is complete. We find that $E(A, -2) = \ker(A + 2I)$ has basis $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$. Since this eigenspace is dimension 2, then $\lambda = -2$ has geometric multiplicity of 2 and is complete. Thus, A is complete because both of its eigenvalues are complete.

(b) Since A is complete, it is diagonalizable. We can show that $E(A, 1) = \ker(A - I)$ is spanned by $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$. So, we see that $A = S\Lambda S^{-1}$ where

$$\begin{aligned}
S &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix}, \\
\Lambda &= \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ and} \\
S^{-1} &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix}.
\end{aligned}$$

So,

$$\begin{aligned}
A^{112} &= S\Lambda^{112}S^{-1} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} (-2)^{112} & 0 & 0 \\ 0 & (-2)^{112} & 0 \\ 0 & 0 & 1^{112} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 2^{112} & 0 & 0 \\ 2^{112} - 1 & 2^{112} & 2^{112} - 1 \\ -2^{112} + 1 & 0 & 1 \end{bmatrix}.
\end{aligned}$$

(c) Again, we use the diagonalization for A:

$$
\begin{aligned}
e^{tA} &= Se^{t\Lambda}S^{-1} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} e^{-2t} & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{t} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix} \\
&= \begin{bmatrix} e^{-2t} & 0 & 0 \\ e^{-2t} - e^{t} & e^{-2t} & e^{-2t} - e^{t} \\ e^{t} - e^{-2t} & 0 & e^{t} \end{bmatrix}.
\end{aligned}
$$

4. (22 points) Consider the following data:

t	1	3	4	7
y	2	0	-3	1

(a) Find the line that best fits the data in the least-squares sense, where we assume that t is the predictor.

(b) Use your line from (a) to predict the value of y when $t = 5$. (Provide an exact answer.)

Solution:

(a) We will find a model of the form $y = \alpha + \beta t$. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 1 & 4 \\ 1 & 7 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ -3 \\ 1 \end{bmatrix}$.

To determine our best-fit line, we need to find the least-squares solution of $A\mathbf{x} = \mathbf{b}$. That is, we need the solution of $A^TA\mathbf{x} = A^T\mathbf{b}$. Note that we will have a unique least-squares solution because the columns of A are linearly independent. We have

$$
A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 4 & 7 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 1 & 4 \\ 1 & 7 \end{bmatrix} = \begin{bmatrix} 4 & 15 \\ 15 & 75 \end{bmatrix}
$$

and

$$
A^T\mathbf{b} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 4 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \end{bmatrix}.
$$

So, we need to solve

$$
\begin{bmatrix} 4 & 15 \\ 15 & 75 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ -3 \end{bmatrix}.
$$

This yields a solution of

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \mathbf{x} = \frac{1}{25} \begin{bmatrix} 15 \\ -4 \end{bmatrix}.$$

So, the least-squares fit line is

$$y = \frac{15 - 4t}{25}.$$

(b) Using our solution from (a), we see that when $t = 5$ we have

$$y = \frac{15 - 4(5)}{25} = -\frac{1}{5}.$$