Write your name below. This exam is worth 100 points. On each problem, you must show all your work to receive credit on that problem. You may have one page of notes to use on this exam. You are not allowed to collaborate on the exam or seek outside help, nor can you use any other notes, the book, the recorded lectures, a calculator, any computational software, or material you find online.

Name:

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1. (32 points: 8 each) If the statement is **always true** mark "TRUE" and provide a *brief* justification; if it is possible for the statement to be false then mark "FALSE" and provide a counterexample.

   (a) $A^T\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is orthogonal to $\ker(A)$.
   Solution:

   True. By the Fredholm alternative, we know that there is a solution for $A^T\mathbf{x} = \mathbf{b}$ if and only if $\mathbf{b}$ is orthogonal to $\operatorname{coker}(A^T)$. But, $\operatorname{coker}(A^T) = \ker(A)$.

   (b) An orthogonal matrix is a square matrix whose columns are orthogonal.
   Solution:

   False. $Q = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ has orthogonal columns, but $Q^TQ = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.

(c) Consider the complex vector space $\mathbb{C}^n$. For $\mathbf{z_1}, \mathbf{z_2} \in \mathbb{C}^n$, $\langle \mathbf{z_1}, \mathbf{z_2} \rangle = \mathbf{z_1}^T \mathbf{z_2}$ defines an inner product.
Solution:

False. Consider $\mathbf{z} = \begin{bmatrix} 1 \\ i \end{bmatrix}$. We have $\mathbf{z}^T \mathbf{z} = 0$. Since $\mathbf{z} \neq \mathbf{0}$, we see that $\langle \mathbf{z_1}, \mathbf{z_2} \rangle$ violates positivity.

(d) Consider the linear function $L : \mathbb{R}^n \to \mathbb{R}^n$. Suppose we have matrix A such that $L[\mathbf{x}] = A\mathbf{x}$. Further, suppose we have a basis $\mathcal{C}$ of $\mathbb{R}^n$ and matrix B where $[L[\mathbf{x}]]_{\mathcal{C}} = B[\mathbf{x}]_{\mathcal{C}}$. True or false: $\det(A) = \det(B)$.
Solution:

True. Let S be the change of basis matrix from $\mathcal{C}$ to the standard basis: $\{\mathbf{e_1}, ..., \mathbf{e_n}\}$. Then, $B = S^{-1}AS$. So,

$$\begin{aligned}
\det(B) &= \det(S^{-1}AS) \\
&= \det(S)\det(A)\det(S^{-1}) \\
&= \frac{\det(S)\det(A)}{\det(S)} \\
&= \det(A).
\end{aligned}$$

2. (18 points) Consider $A = \begin{bmatrix} 1 & -3 \\ 1 & 1 \\ -1 & 1 \end{bmatrix}$

   (a) Find the QR-factorization of A.
   Solution:

   Label the columns of A as $\mathbf{a_1}$ and $\mathbf{a_2}$. We then apply the Gram-Schmidt algorithm to these vectors to obtain an orthogonal basis for img(A):

   $$v_1 = \mathbf{a_1} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \text{ and } \mathbf{v_2} = \mathbf{a_2} - \frac{\langle \mathbf{a_2}, \mathbf{v_1} \rangle}{||\mathbf{v_1}||^2}\mathbf{v_1} = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}.$$

   If we normalize these columns and plug them into a matrix, then we have

   $$Q = \begin{bmatrix} \frac{\sqrt{3}}{3} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{3}}{3} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{3}}{3} & 0 \end{bmatrix}.$$

   We then find $R = Q^T A = \begin{bmatrix} \sqrt{3} & -\sqrt{3} \\ 0 & 2\sqrt{2} \end{bmatrix}$ where $A = QR$.

   (b) Use the QR-factorization from A to solve $A\mathbf{x} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$. (Other methods of solving the linear system will earn no credit.)
   Solution:

   Recall that we can manipulate the equation in the following way:

   $$\begin{aligned} A\mathbf{x} &= \mathbf{b} \\ QR\mathbf{x} &= \mathbf{b} \\ R\mathbf{x} &= Q^T\mathbf{b} \end{aligned}$$

   So, if we solve $R\mathbf{x} = Q^T\mathbf{b} = \begin{bmatrix} \sqrt{3} \\ \sqrt{2} \end{bmatrix}$ by back solving, we see that $\mathbf{x} = \begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \end{bmatrix}$.

3. (16 points) For each of the following, prove or disprove that the following is a linear function.

(a) $L : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2} : A \to A^T$. (That is, the map that sends an 2×2 matrix to its transpose.)
Solution:

This **is** a linear function. To see this, we will show that L preserves both vector addition and scalar multiplication. Note that the vectors in this case are 2×2 matrices. Choose $A, B \in \mathbb{R}^{2\times 2}$ and $c \in \mathbb{R}$. Then, we note that

$$L(A + B) = (A + B)^T = A^T + B^T = L(A) + L(B),$$

and

$$L(cA) = (cA)^T = cA^T = cL(A).$$

We have shown what was needed.

(b) $L : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2} : A \to A^2$. (That is, the map that sends an 2×2 matrix to its square.)
Solution:

This is **not** a linear function. To see this, consider $c = 2$ and $A = I_2$.. Observe that

$$L(2I_2) = 4I_2$$

but

$$2L(I_2) = 2I_2.$$

Since this function does not preserve scalar multiplication, it cannot be a linear function.

4. (18 points) Consider the matrix

$$Z = \begin{bmatrix} 2 & 3 & 2 & 2 \\ 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

From this, define $W = Z^T Z$.

(a) Is W positive definite or not? (Reminder: Justify all your answers.)
Solution:

Yes. We can use row reduction to see that the columns of Z are linearly independent:

$$Z \rightarrow_{\substack{R_2'=R_2-\frac{1}{2}R_1 \\ R_4'=R_4-R_3}} \begin{bmatrix} 2 & 3 & 2 & 2 \\ 0 & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Since W is a Gram-matrix that is constructed from linearly independent vectors, then we know that W is positive definite.

(b) Prove $||\mathbf{x}|| = \sqrt{\mathbf{x}^T W \mathbf{x}}$ is a norm.
Solution:

From (a), we know that W is positive definite. So, $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T W \mathbf{y}$ must be an inner product. It follows that $||\mathbf{x}|| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = \sqrt{\mathbf{x}^T W \mathbf{x}}$ is the associate norm. norm.

5. (16 points) Recall that in any inner product space $\mathcal{V}$ with inner product $\langle \cdot, \cdot \rangle$, the angle θ between vectors $\mathbf{x}, \mathbf{y} \in \mathcal{V}$ satisfy $\cos\theta = \dfrac{\langle \mathbf{x}, \mathbf{y} \rangle}{||\mathbf{x}||\,||\mathbf{y}||}$. For any such inner product space, prove

$$||\mathbf{x}||^2 + ||\mathbf{y}||^2 - 2||\mathbf{x}||\,||\mathbf{y}||\cos\theta = ||\mathbf{x} - \mathbf{y}||^2.$$

Solution:

$$\begin{aligned}
||\mathbf{x}||^2 + ||\mathbf{y}||^2 - 2||\mathbf{x}||\,||\mathbf{y}||\cos\theta &= ||\mathbf{x}||^2 + ||\mathbf{y}||^2 - 2||\mathbf{x}||\,||\mathbf{y}||\frac{\langle \mathbf{x}, \mathbf{y} \rangle}{||\mathbf{x}||\,||\mathbf{y}||} && \text{(Replacing cosine)} \\
&= ||\mathbf{x}||^2 + ||\mathbf{y}||^2 - 2\langle \mathbf{x}, \mathbf{y} \rangle && \text{(Simplifying)} \\
&= \langle \mathbf{x}, \mathbf{x} \rangle - 2\langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle && \text{(Def. of norm)} \\
&= \langle \mathbf{x} - \mathbf{y}, \mathbf{x} - \mathbf{y} \rangle && \text{(Bilinearity and symmetry)} \\
&= ||\mathbf{x} - \mathbf{y}||^2. && \text{(Def. of norm)}
\end{aligned}$$