Write your name below. This exam is worth 100 points. On each problem, you must show all your work to receive credit on that problem. You may have one page of notes to use on this exam. You are not allowed to collaborate on the exam or seek outside help, nor can you use any other notes, the book, the recorded lectures, a calculator, any computational software, or material you find online.

Name:

1. (32 points: 8 each) If the statement is **always true** mark "TRUE" and provide a *brief* justification; if it is possible for the statement to be false then mark "FALSE" and provide a counterexample.

   (a) $A^T\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is orthogonal to $\ker(A)$.

   (b) An orthogonal matrix is a square matrix whose columns are orthogonal.

(c) Consider the complex vector space $\mathbb{C}^n$. For $\mathbf{z_1}, \mathbf{z_2} \in \mathbb{C}^n$, $\langle \mathbf{z_1}, \mathbf{z_2} \rangle = \mathbf{z_1}^T \mathbf{z_2}$ defines an inner product.

(d) Consider the linear function $L : \mathbb{R}^n \to \mathbb{R}^n$. Suppose we have matrix A such that $L[\mathbf{x}] = A\mathbf{x}$. Further, suppose we have a basis $\mathcal{C}$ of $\mathbb{R}^n$ and matrix B where $[L[\mathbf{x}]]_{\mathcal{C}} = B[\mathbf{x}]_{\mathcal{C}}$. True or false: $\det(A) = \det(B)$.

2. (18 points) Consider $A = \begin{bmatrix} 1 & -3 \\ 1 & 1 \\ -1 & 1 \end{bmatrix}$

   (a) Find the QR-factorization of A.

   (b) Use the QR-factorization from A to solve $A\mathbf{x} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$. (Other methods of solving the linear system will earn no credit.)

3. (16 points) For each of the following, prove or disprove that the following is a linear function.

   (a) $L : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2} : A \to A^T$. (That is, the map that sends an 2×2 matrix to its transpose.)

   (b) $L : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2} : A \to A^2$. (That is, the map that sends an 2×2 matrix to its square.)

4. (18 points) Consider the matrix

$$Z = \begin{bmatrix} 2 & 3 & 2 & 2 \\ 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

From this, define $W = Z^T Z$.

(a) Is W positive definite or not? (Reminder: Justify all your answers.)

(b) Prove $||\mathbf{x}|| = \sqrt{\mathbf{x}^T W \mathbf{x}}$ is a norm.

5. (16 points) Recall that in any inner product space $\mathcal{V}$ with inner product $\langle \cdot, \cdot \rangle$, the angle θ between vectors $\mathbf{x}, \mathbf{y} \in \mathcal{V}$ satisfy $\cos\theta = \dfrac{\langle \mathbf{x}, \mathbf{y} \rangle}{||\mathbf{x}||\,||\mathbf{y}||}$. For any such inner product space, prove

$$||\mathbf{x}||^2 + ||\mathbf{y}||^2 - 2||\mathbf{x}||\,||\mathbf{y}||\cos\theta = ||\mathbf{x} - \mathbf{y}||^2.$$