Write your name below. This exam is worth 100 points. On each problem, you must **show all your work** to receive credit on that problem. You may have one page of notes to use on this exam. You are not allowed to collaborate on the exam or seek outside help, nor can you use any other notes, the book, the recorded lectures, a calculator, any computational software, or material you find online.

Name:

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(1) (32 points: 8 each) If the statement is **always true** mark "TRUE" and provide a *brief* justification; if it is possible for the statement to be false then mark "FALSE" and provide a counterexample.

(a) If A is invertible, then so is A^TA.
Solution:

True. Since A is invertible, then we know that $\det(A) \neq 0$ by the Nonsingular Matrix Theorem. Thus, $\det(A^TA) = \det(A^T)\det(A) = \det(A)^2 \neq 0$, which means A^TA is also invertible, again by the Nonsingular Matrix Theorem.

Alternatively, note that $(A^TA)(A^{-1}(A^{-1})^T) = I$, so we have explicitly shown that A^TA has an inverse.

(b) Let $S \subset \mathbb{R}^3$ be defined by

$$S = \{(x, y, z)^T \in \mathbb{R}^3 \quad | \quad x = y \text{ or } x = z\}$$

i.e. the set of all vectors with either the first entry equals the second, or the first entry equals the third. True or false: S is a subspace of $\mathbb{R}^3$.
Solution:

False. Note that $\mathbf{u} = \begin{bmatrix}1\\1\\0\end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix}1\\0\\1\end{bmatrix}$ both lie in S, but $\mathbf{u}+\mathbf{v} = \begin{bmatrix}2\\1\\1\end{bmatrix}$ does not lie in S. That is, S is not closed under vector addition.

(c) If a square matrix A has all ones on its diagonal, then it is nonsingular.
Solution:

False. A counterexample is $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

(d) If $\mathbf{v}_1, ..., \mathbf{v}_k$ are vectors in a vector space $\mathcal{V}$ that do not span $\mathcal{V}$, then they must be linearly independent.
Solution:

False. Consider the vector space $\mathbb{R}^2$. The vectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix} \in \mathbb{R}^2$ do not span $\mathbb{R}^2$, but they are not linearly independent.

(2) (18 points) For each of the following matrices, determine if the matrix is regular. In each case, find it's LU-factorization if possible. If it is not possible, explain why.

$$K = \begin{bmatrix} 0 & 2 & 1 \\ 2 & 6 & 5 \\ 1 & 4 & 4 \end{bmatrix} \qquad M = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 6 & 1 & -2 \\ 1 & 1 & 4 & 3 \end{bmatrix} \qquad N = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 6 & 1 \\ 1 & 1 & 4 \end{bmatrix}$$

Solution:

We see that K is not regular because to get it into row echelon form, a permuation is needed. Since K is not regular, it does not have an LU-factorization.

We see that M is not regular because it is not a square matrix. Since M is not regular, it does not have an LU-factorization.

We see that N is regular through the following row operations:

$$N \to_{\substack{R_2'=R_2-2R_1 \\ R_3'=R_3-R_1}} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix} \to_{R_3'=R_3+\frac{1}{2}R_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & \frac{5}{2} \end{bmatrix}$$

Since N is regular, it has an LU-factorization, and it must be $N = LU$ where

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -\frac{1}{2} & 1 \end{bmatrix} \qquad \text{and} \qquad U = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & \frac{5}{2} \end{bmatrix}$$

(3) Let $A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 2 & 4 & -2 & -1 \\ -1 & 2 & 8 & -2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

(a) (16 points) Find all solutions of $A\mathbf{x} = \mathbf{b}$. If none exist, justify this.
Solution:

We row reduce $[A|\mathbf{b}]$ to row echelon form:

$$[A|\mathbf{b}] \to_{\substack{R_2'=R_2-2R_1 \\ R_3'=R_3+R_1}} \begin{bmatrix} 1 & 2 & -1 & 0 & | & 1 \\ 0 & 0 & 0 & -1 & | & -3 \\ 0 & 4 & 7 & -2 & | & 1 \end{bmatrix} \to_{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 2 & -1 & 0 & | & 1 \\ 0 & 4 & 7 & -2 & | & 1 \\ 0 & 0 & 0 & -1 & | & -3 \end{bmatrix}$$

We see that x_1, x_2, and x_4 are basic variables, and that x_3 is a free variable. Using back substitution, we see that

$$x_4 = 3$$

$$\begin{aligned} x_2 &= \frac{1}{4}\left(-7x_3 + 2(3) + 1\right) \\ &= -\frac{7}{4}x_3 + \frac{7}{4} \end{aligned}$$

$$\begin{aligned} x_1 &= -2\left(-\frac{7}{4}x_3 + \frac{7}{4}\right) - x_3 + 1 \\ &= \frac{9}{2}x_3 - \frac{5}{2}. \end{aligned}$$

So, the solutions of the linear system are

$$\mathbf{x} = x_3 \begin{bmatrix} \frac{9}{2} \\ -\frac{7}{4} \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{5}{2} \\ \frac{7}{4} \\ 0 \\ 3 \end{bmatrix}.$$

(Problem continues on the next page.)

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(b) (8 points) What are the dimensions of the image, coimage, kernel, and cokernel of A?
Solution:

Note that A is a 3×4 matrix. From our work in (a), we know that $\text{Rank}(A) = 3$. Using the Fundamental Theorem of Linear Algebra, we see that

- $\dim(\text{img}\, A) = \dim(\text{coimg}\, A) = \text{Rank}(A) = 3$
- $\dim(\ker A) = 4 - \text{Rank}(A) = 1$
- $\dim(\text{coker}\, A) = 3 - \text{Rank}(A) = 0$

(c) (8 points) Compute a basis for the kernel of A. *Solution*:

From our work in (a), we know that all solutions of $A\mathbf{x} = \mathbf{0}$ are of the form $\mathbf{x} = x_3 \begin{bmatrix} \frac{9}{2} \\ -\frac{7}{4} \\ 1 \\ 0 \end{bmatrix}$.

So, a basis for $\ker(A)$ is

$$\left\{ \begin{bmatrix} \frac{9}{2} \\ -\frac{7}{4} \\ 1 \\ 0 \end{bmatrix} \right\}.$$

(4) (18 points) Suppose A is an $m \times n$ matrix and B is an $r \times m$ matrix.

(a) Prove $\ker(A) \subseteq \ker(BA)$. (That is, show that if $\mathbf{x} \in \ker(A)$, then $\mathbf{x} \in \ker(BA)$.)
Solution:

Choose $\mathbf{x} \in \ker(A)$. Then, we know that $A\mathbf{x} = \mathbf{0}$. So,

$$\begin{aligned} BA\mathbf{x} &= B(A\mathbf{x}) \\ &= B\mathbf{0} \\ &= \mathbf{0}. \end{aligned}$$

Thus, $\mathbf{x} \in \ker(BA)$, and we have shown that $\ker(A) \subseteq \ker(BA)$.

(b) Provide a counterexample to show that, in general, it is not the case that $\ker(BA) \subseteq \ker(A)$. (That is, determine matrices A and B such that there is $\mathbf{x} \in \ker(BA)$ where $\mathbf{x} \notin \ker(A)$.)
Solution:

Consider $A = I_n$, the $n \times n$ identity matrix, and $B = 0_n$, the $n \times n$ zero matrix. Since A is nonsingular, then $\ker(A) = \{\mathbf{0}\}$. But, $BA = 0_n$, so $\ker(BA) = \mathbb{R}^n$ because any vector in $\mathbb{R}^n$ is a solution of $0_n\mathbf{x} = \mathbf{0}$. Since $\mathbb{R}^n \not\subseteq \{\mathbf{0}\}$, then we have that $\ker(BA) \not\subseteq \ker(A)$.