Write your name below. This exam is worth 100 points. On each problem, you must **show all your work** to receive credit on that problem. You may have one page of notes to use on this exam. You are not allowed to collaborate on the exam or seek outside help, nor can you use any other notes, the book, the recorded lectures, a calculator, any computational software, or material you find online.

Name:

(1) (32 points: 8 each) If the statement is **always true** mark "TRUE" and provide a *brief* justification; if it is possible for the statement to be false then mark "FALSE" and provide a counterexample.

(a) If A is invertible, then so is $A^T A$.

(b) Let $S \subset \mathbb{R}^3$ be defined by

$$S = \{(x, y, z)^T \in \mathbb{R}^3 \quad | \quad x = y \text{ or } x = z\}$$

i.e. the set of all vectors with either the first entry equals the second, or the first entry equals the third. True or false: S is a subspace of $\mathbb{R}^3$.

(c) If a square matrix A has all ones on its diagonal, then it is nonsingular.

(d) If $\mathbf{v}_1, ..., \mathbf{v}_k$ are vectors in a vector space $\mathcal{V}$ that do not span $\mathcal{V}$, then they must be linearly independent.

(2) (18 points) For each of the following matrices, determine if the matrix is regular. In each case, find it's LU-factorization if possible. If it is not possible, explain why.

$$K = \begin{bmatrix} 0 & 2 & 1 \\ 2 & 6 & 5 \\ 1 & 4 & 4 \end{bmatrix} \qquad M = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 6 & 1 & -2 \\ 1 & 1 & 4 & 3 \end{bmatrix} \qquad N = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 6 & 1 \\ 1 & 1 & 4 \end{bmatrix}$$

(3) Let $A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 2 & 4 & -2 & -1 \\ -1 & 2 & 8 & -2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

(a) (16 points) Find all solutions of $A\mathbf{x} = \mathbf{b}$. If none exist, justify this.

(Problem continues on the next page.)

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(b) (8 points) What are the dimensions of the image, coimage, kernel, and cokernel of A?

(c) (8 points) Compute a basis for the kernel of A.

(4) (18 points) Suppose A is an $m \times n$ matrix and B is an $r \times m$ matrix.

(a) Prove $\ker(A) \subseteq \ker(BA)$. (That is, show that if $\mathbf{x} \in \ker(A)$, then $\mathbf{x} \in \ker(BA)$.)

(b) Provide a counterexample to show that, in general, it is not the case that $\ker(BA) \subseteq \ker(A)$. (That is, determine matrices A and B such that there is $\mathbf{x} \in \ker(BA)$ where $\mathbf{x} \notin \ker(A)$.)