This exam is worth 100 points. On each problem, you must show all your work to receive credit on that problem. You may have one page of notes to use on this exam. You are not allowed to collaborate on the exam or seek outside help, nor can you use any other notes, the book, the recorded lectures, a calculator, any computational software, or material you find online.

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(1) (40 points: 8 each) If the statement is **always true** mark "TRUE" and provide a *brief* justification; if it is possible for the statement to be false then mark "FALSE" and either provide a *brief* justification or provide a counterexample.

(a) The determinant of a Householder reflection matrix is 1.

Solution: **False.** Consider the unit vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The corresponding Householder reflection matrix is $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ which has determinant -1.

(b) A singular value of $B = \begin{bmatrix} 0 & -1 \\ 2 & 3 \end{bmatrix}$ is 2.

Solution: **False.** The statement will only be true if $2^2 = 4$ is an eigenvalue of $B^T B$. We see that $B^T B = \begin{bmatrix} 4 & 6 \\ 6 & 10 \end{bmatrix}$. Solving $(B^T B - 4I)\mathbf{x} = \begin{bmatrix} 0 & 6 \\ 6 & 6 \end{bmatrix} \mathbf{x} = \mathbf{0}$, we only get $\mathbf{x} = \mathbf{0}$ as a solution. Since 4 is not an eigenvalue of $B^T B$, then $\sqrt{4} = 2$ is not a singular value of B.

(c) If a matrix has only one eigenvalue, then that matrix is complete (diagonalizable).

Solution: **False.** The matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ has only one eigenvalue, $\lambda = 1$. But, we can show that $E(A, 1) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$. That is, A has one eigenvalue, but A is not complete because that one eigenvalue is not complete.

(d) If a square matrix is singular, then 0 is an eigenvalue of that matrix.

Solution: **True.** Suppose A is singular. Then, we know that there exists nonzero $\mathbf{v} \in \ker A$. So, $A\mathbf{v} = 0\mathbf{v}$, which means 0 is an eigenvalue of A.

(e) Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear function where $T \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$. Then, $T \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Solution: **False.** Since T is linear, then we know that

$$T \begin{bmatrix} 2 \\ 2 \end{bmatrix} = T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = T \begin{bmatrix} 1 \\ 2 \end{bmatrix} + T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

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(2) (20 points) Consider $\mathcal{W} = \text{Span}\left\{\begin{bmatrix}4\\2\\1\\2\end{bmatrix}, \begin{bmatrix}-2\\3\\-2\\2\end{bmatrix}\right\}$. Find the the closest point in $\mathcal{W}$ to $\begin{bmatrix}1\\-4\\-1\\2\end{bmatrix}$. (You do not need to simplify your final answer.)

Solution: Note that $\left\{\begin{bmatrix}4\\2\\1\\2\end{bmatrix}, \begin{bmatrix}-2\\3\\-2\\2\end{bmatrix}\right\}$ is a linearly independent set of vectors. So, let $A = \begin{bmatrix}4 & -2\\2 & 3\\1 & -2\\2 & 2\end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix}1\\-4\\-1\\2\end{bmatrix}$. To find the closest point in $\mathcal{W}$ to $\mathbf{b}$, we need to solve

$$A^TA\mathbf{x} = A^T\mathbf{b}.$$

Then, the closest point will be $\mathbf{w} = A\mathbf{x}$. That is, $\mathbf{w} = A(A^TA)^{-1}A^T\mathbf{b}$.
We see that $A^TA = \begin{bmatrix}25 & 0\\0 & 21\end{bmatrix}$, so $(A^TA)^{-1} = \begin{bmatrix}\frac{1}{25} & 0\\0 & \frac{1}{21}\end{bmatrix}$. Also, $A^T\mathbf{b} = \begin{bmatrix}-1\\-8\end{bmatrix}$. So,

$$\mathbf{x} = (A^TA)^{-1}A^T\mathbf{b} = \begin{bmatrix}-\frac{1}{25}\\-\frac{8}{21}\end{bmatrix},$$

which means

$$\mathbf{w} = A\mathbf{x} = \begin{bmatrix}-\frac{4}{25} + \frac{16}{21}\\-\frac{2}{25} - \frac{24}{21}\\-\frac{1}{25} + \frac{16}{21}\\-\frac{2}{25} - \frac{16}{21}\end{bmatrix}.$$

Orthogonal projection can also be used to find the same solution.

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(3) (20 points) Determine a symmetric matrix with eigenvalue 2 that has corresponding eigenvector $\begin{bmatrix}4\\3\end{bmatrix}$ and eigenvalue -6 that has corresponding eigenvector $\begin{bmatrix}-3\\4\end{bmatrix}$.

Solution: We normalize the given eigenvectors and let $Q = \frac{1}{5}\begin{bmatrix}4 & -3\\3 & 4\end{bmatrix}$ and $\Lambda = \begin{bmatrix}2 & 0\\0 & -6\end{bmatrix}$. Then, the matrix that satisfies the given criteria is

$$A = Q\Lambda Q^T = \frac{1}{25}\begin{bmatrix}-22 & 96\\96 & -78\end{bmatrix}.$$

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(4) (20 points) Determine if $p(x, y) = 2x^2 + 2xy + 2y^2 + 4x + 2y - 3$ has a unique absolute minimum value. If it does, determine the minimizer and the absolute minimum value.

Solution: We can rewrite the quadratic function as

$$p(\mathbf{x}) = \mathbf{x}^T K \mathbf{x} - 2\mathbf{x}^T \mathbf{f} + c$$

where $K = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $\mathbf{f} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$ and $c = -3$. We see that K is symmetric, and that

$$K \rightarrow^{R_2' = R_2 - \frac{1}{2}R_1} \begin{bmatrix} 2 & 1 \\ 0 & \frac{3}{2} \end{bmatrix}.$$

That is, K is symmetric, regular, and has positive pivots. So, K is positive definite. This means $p(\mathbf{x})$ has a unique absolute minimum value.

The minimizer is

$$\mathbf{x}^* = K^{-1}\mathbf{f} = \begin{bmatrix} -1 \\ 0 \end{bmatrix},$$

and the absolute minimum is $p(\mathbf{x}^*) = -5$.
