1. (28 points: 7 each) If the statement is **always true** mark "TRUE" and provide a *brief* justification; if it is possible for the statement to be false then mark "FALSE" and provide a counterexample.

(a) Suppose H is the Householder matrix corresponding to unit vector $\mathbf{u}$. Then, $H\mathbf{u} = -\mathbf{u}$.

Solution:
True. We know that $H = I - 2\mathbf{u}\mathbf{u}^T$ and $1 = ||\mathbf{u}||_2 = \mathbf{u}^T\mathbf{u}$. So,

$$\begin{aligned} H\mathbf{u} &= (I - 2\mathbf{u}\mathbf{u}^T)\mathbf{u} \\ &= \mathbf{u} - 2\mathbf{u}\mathbf{u}^T\mathbf{u} \\ &= \mathbf{u} - 2\mathbf{u} \\ &= -\mathbf{u}. \end{aligned}$$

(b) The 1-norm on $\mathbb{R}^n$ is associated with an inner product. That is, there exists an inner product such that $||\mathbf{x}||_1 = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$.

Solution:
False. If a norm is associated with an inner-product, then

$$||\mathbf{x} + \mathbf{y}||^2 + ||\mathbf{x} - \mathbf{y}||^2 = 2(||\mathbf{x}||^2 + ||\mathbf{y}||^2)$$

for each $\mathbf{x}$ and $\mathbf{y}$. So, consider $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. We see that

$$||\mathbf{x} + \mathbf{y}||_1^2 + ||\mathbf{x} - \mathbf{y}||_1^2 = 5^2 + 1^2 = 26$$

and

$$2(||\mathbf{x}||_1^2 + ||\mathbf{y}||_1^2) = 2(2^2 + 1^2) = 10.$$

(c) If $\{\mathbf{v}_1, ..., \mathbf{v}_n\} \subseteq \mathbb{R}^n$ is an orthogonal set of vectors, then $A = [\mathbf{v}_1 \cdots \mathbf{v}_n]$ is an orthogonal matrix.

Solution:

False. The columns of $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ are orthogonal, but $A^T A = 2I$, so A is not orthogonal.

(d) If $\mathcal{V}$ has inner-product $\langle \cdot, \cdot \rangle$ with associated norm $||\cdot||$, then $||\mathbf{x}+\mathbf{y}||^2 = ||\mathbf{x}||^2 + ||\mathbf{y}||^2$ whenever $\mathbf{x}$ and $\mathbf{y}$ are orthogonal.

Solution:

True. Suppose $\mathcal{V}$ has inner-product $\langle \cdot, \cdot \rangle$ with associated norm $||\cdot||$ and that $\mathbf{x}$ and $\mathbf{y}$ are orthogonal. Then,

$$\begin{aligned} ||\mathbf{x}+\mathbf{y}||^2 &= \langle \mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y} \rangle \\ &= \langle \mathbf{x}, \mathbf{x} \rangle + 2\langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle \\ &= \langle \mathbf{x}, \mathbf{x} \rangle + 2 \cdot 0 + \langle \mathbf{y}, \mathbf{y} \rangle \\ &= \langle \mathbf{x}, \mathbf{x} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle \\ &= ||\mathbf{x}||^2 + ||\mathbf{y}||^2. \end{aligned}$$

2. (21 points) For each of the following matrices, determine if it is positive definite. If it is, determine its Cholesky Factorization.

(a) $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}$

(b) $B = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}$

(c) $C = \begin{bmatrix} 1 & -3 & 1 \\ 2 & -2 & 1 \\ 3 & 7 & 5 \end{bmatrix}$

(a) We see that A is symmetric. Row reducing, we see that

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} \to^{R_2' = R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \to^{R_3' = R_3 - R_2} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 2 \end{bmatrix}.$$

So, A is symmetric and regular, but it does not have positive pivots when reduced to REF using only row operations of type 1. So, A is not positive definite.

(b) We see that B is symmetric. Row reducing, we see that

$$B = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} \to^{R_2'=R_2-\frac{1}{2}R_1} \begin{bmatrix} 2 & 1 & 0 \\ 0 & \frac{5}{2} & -1 \\ 0 & -1 & 1 \end{bmatrix} \to^{R_3'=R_3+\frac{2}{5}R_2} \begin{bmatrix} 2 & 1 & 0 \\ 0 & \frac{5}{2} & -1 \\ 0 & 0 & \frac{3}{5} \end{bmatrix} = U.$$

Since B is symmetric, regular, and has positive pivots when reduced to REF using only row operations of type 1, then B is positive definite.

We set out to find the Cholesky factor of B. With U as defined above, we have

$$U = DL^T = \begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{5}{2} & 0 \\ 0 & 0 & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{2}{5} \\ 0 & 0 & 1 \end{bmatrix}.$$

So, the Cholesky factor is

$$G = LD^{\frac{1}{2}} = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ \frac{\sqrt{2}}{2} & \sqrt{\frac{5}{2}} & 0 \\ 0 & -\frac{2\sqrt{5}}{5\sqrt{2}} & \sqrt{\frac{3}{5}} \end{bmatrix}.$$

(c) We see that C is not symmetric, so C is not positive definite.

3. (21 points) Consider $A = \begin{bmatrix} 3 & 2 & 5 \\ 6 & 1 & 1 \\ 6 & -2 & 10 \end{bmatrix}$.

(a) Determine the Gram-Schmidt QR-Factorization of A.

(b) Use the Gram-Schmidt QR-Factorization of A to solve $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b} = \begin{bmatrix} 24 \\ 42 \\ 54 \end{bmatrix}$. (Other methods of solving $A\mathbf{x} = \mathbf{b}$ will earn no credit here.)

(a) We apply the Gram-Schmidt algorithm to the columns of A and then normalize; this will provide the columns of Q.

Write $A = [\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3]$. Then, the Gram-Schmidt algorithm gives

$$\mathbf{v}_1 = \mathbf{a}_1 = \begin{bmatrix} 3 \\ 6 \\ 6 \end{bmatrix}$$

$$\mathbf{v}_2 = \mathbf{a}_2 - \frac{\mathbf{a}_2^T \mathbf{v}_1}{\mathbf{v}_1^T \mathbf{v}_1} \mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

$$\mathbf{v}_3 = \mathbf{a}_3 - \frac{\mathbf{a}_3^T \mathbf{v}_1}{\mathbf{v}_1^T \mathbf{v}_1} \mathbf{v}_1 - \frac{\mathbf{a}_3^T \mathbf{v}_2}{\mathbf{v}_2^T \mathbf{v}_2} \mathbf{v}_2 = \begin{bmatrix} 4 \\ -4 \\ 2 \end{bmatrix}.$$

We can then normalize to get the orthonormal basis

$$\left\{ \mathbf{u}_1 = \frac{1}{3} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \mathbf{u}_2 = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \mathbf{u}_3 = \frac{1}{3} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \right\}.$$

So,

$$Q = [\mathbf{u}_1 \mathbf{u}_2 \mathbf{u}_3] = \frac{1}{3} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}.$$

From this, we see that

$$R = Q^T A = \begin{bmatrix} 9 & 0 & 9 \\ 0 & 3 & -3 \\ 0 & 0 & 6 \end{bmatrix},$$

where $A = QR$.

(b) We have

$$\mathbf{b} = A\mathbf{x} = QR\mathbf{x},$$

which implies we need only solve

$$R\mathbf{x} = Q^T \mathbf{b} = \begin{bmatrix} 72 \\ -6 \\ 6 \end{bmatrix}.$$

Since R is upper triangular, we need only use back substitution to see that

$$\mathbf{x} = \begin{bmatrix} 7 \\ -1 \\ 1 \end{bmatrix}.$$

4. (30 points) Consider the vector space $C^0[0,1]$, the space of continuous real-valued functions with domain $[0,1]$. Throughout this problem, make use of the inner-product $\langle f, g\rangle = \int_0^1 f(x)g(x)e^x dx$.

   You may use the following values in this problem without justification:

$$\int_0^1 xe^x dx = 1$$
$$\int_0^1 x^2 e^x dx = e - 2$$
$$\int_0^1 x^3 e^x dx = 6 - 2e$$
$$\int_0^1 x^4 e^x dx = 9e - 24$$
$$\int_0^1 x^5 e^x dx = 120 - 44e$$

   Complete the following:

   (a) Prove $\langle f, g\rangle$ is an inner product on $C^0[0,1]$.

   (b) Determine the angle between $f(x) = x$ and $g(x) = 1$. (Provide an exact expression for your answer, not a decimal approximation.)

   (c) Consider the subspace $\mathcal{W} = \text{Span}\{1, x\}$ of $C^0[0,1]$. Determine an orthogonal basis for this subspace.

   (d) Find the orthogonal projection of x^2 onto $\mathcal{W} = \text{Span}\{1, x\}$. (Write your answer as a linear combination of an appropriate basis where the coefficients are simplified.)

Solution:

(a) We check each of the axioms. Suppose $f, g, h \in C^0[0,1]$ and $c, d \in \mathbb{R}$.

- **Bilinearity:**

$$\begin{aligned}\langle cf + dg, h\rangle &= \int_0^1 (cf(x) + dg(x))h(x)e^x dx \\ &= c\int_0^1 f(x)h(x)e^x dx + d\int_0^1 g(x)h(x)e^x dx \\ &= c\langle f, h\rangle + d\langle g, h\rangle.\end{aligned}$$

- **Symmetry:**

$$\begin{aligned}\langle f, g\rangle &= \int_0^1 f(x)g(x)e^x dx \\ &= \int_0^1 g(x)f(x)e^x dx \\ &= \langle g, f\rangle.\end{aligned}$$

- **Positivity** It is clear that

$$\langle f, f, \rangle = \int_0^1 [f(x)]^2 e^x dx \geq 0$$

because $[f(x)]^2 e^x \geq 0$. Further, if $\langle f, f, \rangle = 0$, then $[f(x)]^2 e^x = 0$ (since the integrand is continuous), which means $f(x) = 0$.

(b) Let θ be the angle between $f(x) = x$ and $g(x) = 1$. Then, $\cos\theta = \frac{\langle f,g\rangle}{||f||||g||}$. We note that

$$\langle f, g\rangle = \int_0^1 xe^x dx = 1,$$

$$||f|| = \sqrt{\langle f, f\rangle} = \sqrt{\int_0^1 x^2 e^x dx} = \sqrt{e-2},$$

and

$$||g|| = \sqrt{\langle g, g\rangle} = \sqrt{\int_0^1 e^x dx} = \sqrt{e-1}.$$

So, $\cos\theta = \frac{1}{\sqrt{(e-1)(e-2)}}$, which means

$$\theta = \arccos\left(\frac{1}{\sqrt{(e-1)(e-2)}}\right).$$

(c) We know that $\{1, x\}$ is not itself an orthogonal basis because we showed in (b) that the angle between 1 and x is not $\frac{\pi}{2}$. So, we will apply the Gram-Schmidt algorithm to these vectors:

$$v_1 = 1$$

$$\begin{aligned}v_2 &= x - \frac{\langle x, 1\rangle}{||1||^2} \cdot 1 \\ &= x - \frac{1}{e-1}.\end{aligned}$$

So, an orthogonal basis for $\mathcal{W}$ is $\{1, x - \frac{1}{e-1}\}$.

(d) We will use our basis from (c). The orthogonal projection is

$$\begin{aligned} w &= \frac{\langle x^2, 1\rangle}{||1||^2}\cdot 1 + \frac{\langle x^2, x-\frac{1}{e-1}\rangle}{||x-\frac{1}{e-1}||^2}\left(x-\frac{1}{e-1}\right) \\ &= \frac{e-2}{e-1} + \frac{\langle x^2, x\rangle - \frac{1}{e-1}\langle x^2, 1\rangle}{\langle x, x\rangle - \frac{2}{e-1}\langle x, 1\rangle + \frac{1}{(e-1)^2}\langle 1, 1\rangle}\left(x-\frac{1}{e-1}\right) \\ &= \frac{e-2}{e-1} + \frac{6-2e-\frac{e-2}{e-1}}{e-2-\frac{2}{e-1}+\frac{1}{e-1}}\left(x-\frac{1}{e-1}\right) \\ &= \frac{e-2}{e-1} + \frac{-2e^2+7e-4}{e^2-3e+1}\left(x-\frac{1}{e-1}\right). \end{aligned}$$