This exam is worth 100 points. On each problem, you must show all your work to receive credit on that problem. This exam is due to Gradescope on July 8 at 11:59pm (MDT). You may use your notes, textbook, and any other resource provided on the course webpage. You are not allowed to collaborate on the exam or seek outside help, nor can you use a calculator, any computational software, or other materials posted online.

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1. (28 points: 7 each) If the statement is **always true** mark "TRUE" and provide a *brief* justification; if it is possible for the statement to be false then mark "FALSE" and provide a counterexample.

   (a) Suppose H is the Householder matrix corresponding to unit vector $\mathbf{u}$. Then, $H\mathbf{u} = -\mathbf{u}$.

   (b) The 1-norm on $\mathbb{R}^n$ is associated with an inner product. That is, there exists an inner product such that $||\mathbf{x}||_1 = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$.

   (c) If $\{\mathbf{v}_1, ..., \mathbf{v}_n\} \subseteq \mathbb{R}^n$ is an orthogonal set of vectors, then $A = [\mathbf{v}_1 \cdots \mathbf{v}_n]$ is an orthogonal matrix.

   (d) If $\mathcal{V}$ has inner-product $\langle \cdot, \cdot \rangle$ with associated norm $|| \cdot ||$, then $||\mathbf{x} + \mathbf{y}||^2 = ||\mathbf{x}||^2 + ||\mathbf{y}||^2$ whenever $\mathbf{x}$ and $\mathbf{y}$ are orthogonal.

2. (21 points) For each of the following matrices, determine if it is positive definite. If it is, determine its Cholesky Factor.

   (a) $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}$

   (b) $B = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}$

   (c) $C = \begin{bmatrix} 1 & -3 & 1 \\ 2 & -2 & 1 \\ 3 & 7 & 5 \end{bmatrix}$

3. (21 points) Consider $A = \begin{bmatrix} 3 & 2 & 5 \\ 6 & 1 & 1 \\ 6 & -2 & 10 \end{bmatrix}$.

   (a) Determine the Gram-Schmidt QR-Factorization of A.

   (b) Use the Gram-Schmidt QR-Factorization of A to solve $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b} = \begin{bmatrix} 24 \\ 42 \\ 54 \end{bmatrix}$. (Other methods of solving $A\mathbf{x} = \mathbf{b}$ will earn no credit here.)

4. (30 points) Consider the vector space $C^0[0, 1]$, the space of continuous real-valued functions with domain $[0, 1]$. Throughout this problem, make use of the inner-product $\langle f, g \rangle = \int_0^1 f(x)g(x)e^x dx$.

   You may use the following values in this problem without justification:

$$\int_0^1 xe^x dx = 1$$
$$\int_0^1 x^2e^x dx = e - 2$$
$$\int_0^1 x^3e^x dx = 6 - 2e$$
$$\int_0^1 x^4e^x dx = 9e - 24$$
$$\int_0^1 x^5e^x dx = 120 - 44e$$

Complete the following:

(a) Prove $\langle f, g\rangle$ is an inner product on $C^0[0,1]$.

(b) Determine the angle between $f(x) = x$ and $g(x) = 1$. (Provide an exact expression for your answer, not a decimal approximation.)

(c) Consider the subspace $\mathcal{W} = \text{Span}\{1, x\}$ of $C^0[0,1]$. Determine an orthogonal basis for this subspace.

(d) Find the orthogonal projection of x^2 onto $\mathcal{W} = \text{Span}\{1, x\}$. (Write your answer as a linear combination of an appropriate basis where the coefficients are simplified.)