1. (28 points: 7 each) If the statement is **always true** mark "TRUE" and provide a *brief* justification; if it is possible for the statement to be false then mark "FALSE" and provide a counterexample.

   (a) If U is an upper triangular matrix and has an inverse, then its inverse is lower triangular.

   False. As a counterexample, consider the upper triangular matrix $U = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. It's inverse is $U^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$ which is not lower triangular.

   (b) If the rank of A is 2 and A is a 2×4 matrix, then the kernel of A has dimension 2.

   True. From the Fundamental Theorem of Linear Algebra (FTLA), we know that the number of columns of A is the sum of the rank of A and the dimension of the kernal of A. That is, $4 = 2 + \dim(\ker(A))$, which implies $\dim(\ker(A)) = 2$.

(c) If $\det(A) = 0$, then $A\mathbf{x} = \mathbf{b}$ has no solutions.

False. Consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\mathbf{b} = \mathbf{0}$. We have $\det(A) = 0$ but $A\mathbf{x} = \mathbf{b}$ has solution $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(d) If $\mathcal{W}$ is the set of $n \times n$ matrices, A, such that $\det(A) = 0$, then $\mathcal{W}$ a subspace of the $\mathbb{R}^{n \times n}$, the vector space of all $n \times n$ matrices.

False. Note that $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ both lie in $\mathcal{W}$, but $A + B = I$ does not. That is, $\mathcal{W}$ is not closed under addition, so $\mathcal{W}$ cannot be a subspace of $\mathbb{R}^{n \times n}$.

2. (16 points) Consider $A = \begin{bmatrix} 1 & -1 & 2 & 1 \\ 2 & 1 & -2 & -1 \\ 1 & 2 & -4 & -3 \\ 0 & 3 & -6 & -2 \end{bmatrix}$. A is row equivalent to $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and A^T is row equivalent to $\begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Use this information to determine bases for the four fundamental subspaces associated with the matrix A. (Clearly indicate which basis belongs to which subspace.)

The columns of A that correspond to the pivots of its reduced form are a basis for $\text{Ran}(A)$:

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -3 \\ -2 \end{bmatrix} \right\}.$$

The rows of a row echelon form of A form a basis for $\text{Coran}(A)$:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

To find a basis for $\ker(A)$, we consider the solutions of $A\mathbf{x} = \mathbf{0}$. Note that $x_1 = 0$, $x_2 = 2x_3$, and $x_4 = 0$ where x_3 is free. So, a basis for $\ker(A)$ is

$$\left\{ \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

To find a basis for $\text{Coker}(A)$, we consider the solutions of $A^T\mathbf{x} = \mathbf{0}$. Note that $x_1 = 3x_4$, $x_2 = -2x_4$, and $x_3 = x_4$ where x_4 is free. So, a basis for $\text{Coker}(A)$ is

$$\left\{ \begin{bmatrix} 3 \\ -2 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

3. Consider the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 5 & 5 \\ 2 & 1 & 2 \end{bmatrix}$.

   (a) (14 points) Use Gauss-Jordan Elimination to find the inverse of **A**.

We set up the augmented matrix $[A|I]$ and row reduce until we have I on the left-side of the partition.

$$[A|I] = \begin{bmatrix} 1 & 2 & 3 & | & 1 & 0 & 0 \\ 3 & 5 & 5 & | & 0 & 1 & 0 \\ 2 & 1 & 2 & | & 0 & 0 & 1 \end{bmatrix} \rightarrow_{\substack{R_2'=R_2-3R_1 \\ R_3'=R_3-2R_1}} \begin{bmatrix} 1 & 2 & 3 & | & 1 & 0 & 0 \\ 0 & -1 & -4 & | & -3 & 1 & 0 \\ 0 & -3 & -4 & | & -2 & 0 & 1 \end{bmatrix} \rightarrow_{R_2'=-R_2}$$

$$\begin{bmatrix} 1 & 2 & 3 & | & 1 & 0 & 0 \\ 0 & 1 & 4 & | & 3 & -1 & 0 \\ 0 & -3 & -4 & | & -2 & 0 & 1 \end{bmatrix} \rightarrow_{\substack{R_1'=R_1-2R_2 \\ R_3'=R_3+3R_2}} \begin{bmatrix} 1 & 0 & -5 & | & -5 & 2 & 0 \\ 0 & 1 & 4 & | & 3 & -1 & 0 \\ 0 & 0 & 8 & | & 7 & -3 & 1 \end{bmatrix} \rightarrow_{R_3'=\frac{1}{8}R_3}$$

$$\begin{bmatrix} 1 & 0 & -5 & | & -5 & 2 & 0 \\ 0 & 1 & 4 & | & 3 & -1 & 0 \\ 0 & 0 & 1 & | & \frac{7}{8} & \frac{-3}{8} & \frac{1}{8} \end{bmatrix} \rightarrow_{\substack{R_1'=R_1+5R_3 \\ R_2'=R_2-4R_3}} \begin{bmatrix} 1 & 0 & 0 & | & -\frac{5}{8} & \frac{1}{8} & \frac{5}{8} \\ 0 & 1 & 0 & | & -\frac{4}{8} & \frac{4}{8} & -\frac{4}{8} \\ 0 & 0 & 1 & | & \frac{7}{8} & \frac{-3}{8} & \frac{1}{8} \end{bmatrix}$$

So, we have $A^{-1} = \frac{1}{8} \begin{bmatrix} -5 & 1 & 5 \\ -4 & 4 & -4 \\ 7 & -3 & 1 \end{bmatrix}$.

(b) (6 points) Use your answer from (a) to find the solution of

$$\begin{cases} x_1 + 2x_2 + 3x_3 = -3 \\ 3x_1 + 5x_2 + 5x_3 = 0 \\ 2x_1 + x_2 + 2x_3 = 2 \end{cases}.$$

You must use your answer from (a) to receive points. (Other methods will receive no points.)

$$\mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{8}\begin{bmatrix} -5 & 1 & 5 \\ -4 & 4 & -4 \\ 7 & -3 & 1 \end{bmatrix}\begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 25 \\ 4 \\ -19 \end{bmatrix}.$$

4. (15 points) Prove that if A is nonsingular, then $A^T A$ is also nonsingular.

One Possibility:

Recall that a matrix is nonsingular if and only if it is its determinant is nonzero. Also, $\det(A) = \det(A^T)$. So, $\det(A)$ is nonzero and

$$\det(A^T A) = \det(A^T)\det(A) = \det(A)\det(A) = \det(A)^2 \neq 0.$$

Thus, $A^T A$ is nonsingular.

Another Possibility:

Recall that a matrix is nonsingular if and only if it is invertible. Since A is invertible, then we know that A^T is also invertible with inverse $(A^T)^{-1} = (A^{-1})^T$. We observe that

$$\begin{aligned} (A^T A)(A^{-1}(A^{-1})^T) &= A^T(AA^{-1})(A^{-1})^T \\ &= A^T I (A^{-1})^T \\ &= A^T (A^{-1})^T \\ &= I. \end{aligned}$$

We have shown that $A^T A$ is invertible. Therefore it is nonsingular.

5. Let $A = \begin{bmatrix} 0 & 0 & -4 \\ 1 & 2 & 3 \\ 0 & 1 & 7 \end{bmatrix}$.

(a) (15 points) Determine the permuted LU-factorization of A.

$$A \to^{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -4 \\ 0 & 1 & 7 \end{bmatrix} \to^{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 7 \\ 0 & 0 & -4 \end{bmatrix} = U$$

where $L = I$ because no elimination operations were applied, and

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

because of the row interchanges that were applied. With these matrices, we have $PA = LU$.

(b) (6 points) Use the answer from (a) to find the determinant of A. (Other methods will receive no credit here.)

From (a), we know U and that $N = 2$ because two elementary permuations were applied. So,

$$\det(A) = (-1)^N \det(U) = (-1)^2(1)(1)(-4) = -4.$$