The final exam is due on May 5 by 10 pm. As with the previous exams, please upload your work to Gradescope.

From problem 2 onward, you *must* show all your work to receive credit on that problem. You are allowed to use your notes and the book. You are not allowed to collaborate on the exam or seek outside help, nor can you use a calculator, any computational software, or material you find online. You may use computational software to check your answer, but you must provide all work. No decimal answers will be accepted.

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1. (20 points: 2 each) If the statement is **always true** mark "TRUE"; if it is possible for the statement to be false then mark "FALSE." If you think the statement is neither true nor false but just incoherent, contact your instructor. No justification is necessary.

   (a) If square matrix A has a 0 on its main diagonal, then A is not regular.

   (b) If diagonalizable matrix A has characteristic polynomial $p(\lambda)$ then $p(A)$ is the zero matrix. (Multiply any constant terms in p(A) by the appropriate identity matrix.)

   (c) Let A be a symmetric matrix with real entries. If $\ker A = \{0\}$ then $A^2 > 0$.

   (d) If A^+ is the pseudo inverse of A then $AA^+A = A$

   (e) If real, square matrix A has the QR factorization $A = QR$, then QRR^TQ^T is the spectral decomposition of AA^T.

   (f) If A is a 5×5 matrix it must have at least one real eigenvalue.

   (g) Let A be an $n \times n$ matrix. If the system $Ax = 0$ has only the trivial solution, then the system $Ax = b$ (with all sizes being compatible) must have a solution.

   (h) The set of complex vectors of the form $(z, \overline{z})^T$ for $z \in \mathbb{C}$ form a subspace $\mathbb{C}^2$.

   (i) Applying an elementary row operation to an orthogonal matrix gives an orthogonal matrix.

   (j) If Q is a 2×2 improper orthogonal matrix then $Q^2 = I$.

   Solution

   (a) **False,** a counterexample is $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

   (b) **True**, since $A = S\Lambda S^{-1}$, $p(A) = Sp(\Lambda)S^{-1}$. Since Λ is diagonal with the eigenvalues for entries, $p(\Lambda) = 0$, so $p(A) = 0$.

   (c) **True**, since $A^T = A$, we have $A^2 = A^TA$, which is positive definite since the columns of A are linearly independent when $\ker(A) = \{0\}$.

(d) **True**, if $A = P\Sigma Q^T$ then $AA^+A = P\Sigma Q^T Q\Sigma^+ P^T P\Sigma Q^T = P\Sigma\Sigma^+\Sigma Q^T = P\Sigma Q^T$

(e) **False,** RR^T is typically not diagonal, as required by the spectral decomposition.

(f) **True**, for a real matrix since $n = 5$ is odd and complex roots come in pairs, then the characteristic polynomial of A must have at least one real root. Since the questions did not specify that the matrix was real (either answer will be correct).

(g) **True**, we know that A must be invertible.

(h) **False,** it is not closed under scalar multiplication as $i(z, \overline{z}) = (iz, i\overline{z})$, but $\overline{iz} = -i\overline{z}$.

(i) **False,** this is true only when you exchange rows or multiply a row by -1.

(j) **True**, a 2×2 orthogonal matrix has the form $Q = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix}$, which gives that $Q^2 = I$.

2. (16 points) Let $p(\mathbf{x}) = x^2 + 2y^2 + 4z^2 - 2xy + 4xz - 4yz - 2ax - 2bz + c$, where $\mathbf{x} = (x, y, z)$.

   (a) (3 points) Find matrix K and vector $\mathbf{f}$ such that $p(\mathbf{x}) = \mathbf{x}^T K\mathbf{x} - 2\mathbf{x}^T\mathbf{f} + c$.

   (b) (7 points) Using part (a), find conditions on a, b, c such that $p(\mathbf{x})$ has a finite minimum. Your conditions must be as general as possible.

   (c) (6 points) Let a, b, c be fixed and satisfy conditions from part (b), find the minimizer (or a set of minimizers, if it is not unique) and the minimum of $p(\mathbf{x})$.

Solution: (a)

$$K = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 2 & -2 \\ 2 & -2 & 4 \end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix} a \\ 0 \\ b \end{bmatrix}.$$

(b) c does not affect the existence of the minimum, so $c \in \mathbb{R}$. We know that, in general, a finite minimum exists if $K \geq 0$ and $\mathbf{f} \in \operatorname{img} K$. We perform regular Gaussian elimination to check both.

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & a \\ -1 & 2 & -2 & 0 \\ 2 & -2 & 4 & b \end{array}\right] \iff \dots \iff \left[\begin{array}{ccc|c} 1 & 0 & 2 & 2a \\ 0 & 1 & 0 & a \\ 0 & 0 & 0 & b - 2a \end{array}\right].$$

First, we see that $K \geq 0$. Second, for the system to be consistent we must have $b = 2a$. The conditions are as follows: $a, c \in \mathbb{R}$, $b = 2a$, or any equivalent.

(c) Since we have a free variable, there are infinitely many solutions to the system and hence infinitely many minimizers. One of the ways to describe the set is $\{[2a, a, 0]^T + s[-2, 0, 1]^T \mid s \in \mathbb{R}\}$. Denote $\mathbf{x}^* = [2a - 2s, a, s]^T$. Using the formula, the minimum is

$$\begin{aligned} p(\mathbf{x}^*) &= c - \mathbf{f}^T\mathbf{x}^* \\ &= c - [a, 0, 2a] \cdot [2a - 2s, a, s] \\ &= c - a(2a - 2s) - 0a + 2as \\ &= c - 2a^2. \end{aligned}$$

3. (14 points) Consider the linear map $L : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$L(x, y, z) = (x + 3y + 2z, x - 4z, y + 3z)^T.$$

(a) (6 points) Find the matrix representation of L with respect to the standard basis.

Solution: To find the matrix representation of L in the standard basis we just read off the coefficients $A = \begin{pmatrix} 1 & 3 & 2 \\ 1 & 0 & -4 \\ 0 & 1 & 3 \end{pmatrix}$.

(b) (8 points) Find the matrix representation of L with respect to the basis:

$$S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

Solution: The change of basis matrix is $P = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ and inverse $P^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix}$.

The matrix representation with respect to the basis S is given by:

$$B = P^{-1}AP = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 3 & 2 \\ 1 & 0 & -4 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 1 & 0 \\ -7 & 0 & 1 \\ 9 & 3 & 0 \end{pmatrix}.$$

4. (16 points) The parts of this problem are related.

   (a) (8 points) Find a symmetric matrix, A, that has eigenvalues $\lambda_1 = 1, \lambda_2 = 10, \lambda_3 = 0$ with respective eigenvectors $v_1 = (1,2,0), v_2 = (2,-1,0)$, and $v_3 = (0,0,1)$.

   Solution: We seek a symmetric matrix, so we want to use the spectral theorem. For this, we can use the eigenvectors to construct an orthonormal basis. Note, the eigenvectors are orthogonal, but not normal. Thus, we have to normalize to get the matrix $Q = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and

$$A = QDQ^T = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{41}{5} & -\frac{18}{5} & 0 \\ -\frac{18}{5} & \frac{14}{5} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

   (b) (8 points) Find e^A. **Solution:** Since we have a diagonalization for A we can compute as follows:

$$e^A = Qe^DQ = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^1 & 0 & 0 \\ 0 & e^{10} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{e^1+4e^{10}}{5} & \frac{2e^1-2e^{10}}{5} & 0 \\ \frac{2e^1-2e^{10}}{5} & \frac{4e^1+e^{10}}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

5. (18 points) Let $A = \begin{pmatrix} 0 & 1 & 0 \\ -4 & 4 & 0 \\ -2 & 2 & 1 \end{pmatrix}$

   (a) (6 points) Find the eigenvalues of A and their multiplicities.

   (b) (4 points) What is J, the Jordan Canonical Form of A?

   (c) (8 points) Find the matrix S such that $A = SJS^{-1}$.

Solution:

(a) $\det(A - \lambda I) = (1-\lambda)\det\begin{pmatrix} -\lambda & 1 \\ -4 & 4-\lambda \end{pmatrix} = (1-\lambda)(2-\lambda)^2$

Our eigenvalues are 1 with algebraic multiplicity 1 and 2 with algebraic multiplicity 2. $\lambda = 1$ will have geometric multiplicity of 1. For $\lambda = 2$ we see that it only has a single eigenvector:

$$A - 2I = \begin{pmatrix} -2 & 1 & 0 \\ -4 & 2 & 0 \\ -2 & 2 & -1 \end{pmatrix} \to \begin{pmatrix} -2 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$v_2 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

So $\lambda = 2$ has geometric multiplicty 1 and is not a complete eigenvalue.

(b) $J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

(c) We need the eigenvector for $\lambda = 1$ and the generalized eigenvector for $\lambda = 2$ to proceed.
$\lambda = 1$:

$$A - I = \begin{pmatrix} -1 & 1 & 0 \\ -4 & 3 & 0 \\ -2 & 2 & 0 \end{pmatrix} \to \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$v_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

To find the generalized eigenvector for $\lambda = 2$ we solve the augmented matrix:

$$\left(\begin{array}{ccc|c} -2 & 1 & 0 & 1 \\ -4 & 2 & 0 & 2 \\ -2 & 2 & -1 & 2 \end{array}\right) \to \left(\begin{array}{ccc|c} -2 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$w = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

So our matrix is

$$S = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 2 & 0 \end{pmatrix}$$

6. (16 points) Let A be the matrix with the SVD given by

$$A = \begin{pmatrix} 3/5 & 0 \\ 0 & 1 \\ 4/5 & 0 \end{pmatrix} \begin{pmatrix} 15 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1/3 & 2/3 & -2/3 \\ -2/3 & 2/3 & 1/3 \end{pmatrix}$$

(a) (4 points) What is the rank of A? Does A have an inverse?

(b) (6 points) Find the best rank 1 approximation for A.

(c) (6 points) Find the pseudoinverse of A and use it to find the least squares solution to $Ax = b$ where $b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Solution:

(a) The rank of A is the number of singular values, so $rank A = 2$. Since A is a 3×3 matrix, it is singular and does not have an inverse.

(b) The best rank 1 approximation of A is

$$\begin{pmatrix} 3/5 \\ 0 \\ 4/5 \end{pmatrix} 15 \begin{pmatrix} 1/3 & 2/3 & -2/3 \end{pmatrix} = \begin{pmatrix} 3 & 6 & -6 \\ 0 & 0 & 0 \\ 4 & 8 & -8 \end{pmatrix}$$

(c) The pseudoinverse is given by

$$A^+ = \begin{pmatrix} 1/3 & -2/3 \\ 2/3 & 2/3 \\ -2/3 & 1/3 \end{pmatrix} \begin{pmatrix} 1/15 & 0 \\ 0 & 1/3 \end{pmatrix} \begin{pmatrix} 3/5 & 0 & 4/5 \\ 0 & 1 & 0 \end{pmatrix}$$

$$A^+ = \frac{1}{15^2} \begin{pmatrix} 3 & -50 & 4 \\ 6 & 50 & 8 \\ -6 & 25 & -8 \end{pmatrix}$$

And the least squares solution to $Ax = b$ is

$$x^* = A^+ b = \frac{1}{15^2} \begin{pmatrix} 3 & -50 & 4 \\ 6 & 50 & 8 \\ -6 & 25 & -8 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{15^2} \begin{pmatrix} -43 \\ 64 \\ 11 \end{pmatrix}$$