The final exam is due on May 5 by 10 pm. As with the previous exams, please upload your work to Gradescope.

From problem 2 onward, you *must* show all your work to receive credit on that problem. You are allowed to use your notes and the book. You are not allowed to collaborate on the exam or seek outside help, nor can you use a calculator, any computational software, or material you find online. You may use computational software to check your answer, but you must provide all work. No decimal answers will be accepted.

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1. (20 points: 2 each) If the statement is **always true** mark "TRUE"; if it is possible for the statement to be false then mark "FALSE." If you think the statement is neither true nor false but just incoherent, contact your instructor. No justification is necessary.

   (a) If square matrix A has a 0 on its main diagonal, then A is not regular.

   (b) If diagonalizable matrix A has characteristic polynomial $p(\lambda)$ then $p(A)$ is the zero matrix. (Multiply any constant terms in p(A) by the appropriate identity matrix.)

   (c) Let A be a symmetric matrix with real entries. If $\ker A = \{0\}$ then $A^2 > 0$.

   (d) If A^+ is the pseudo inverse of A then $AA^+A = A$

   (e) If real, square matrix A has the QR factorization $A = QR$, then QRR^TQ^T is the spectral decomposition of AA^T.

   (f) If A is a 5×5 matrix it must have at least one real eigenvalue.

   (g) Let A be an $n \times n$ matrix. If the system $Ax = 0$ has only the trivial solution, then the system $Ax = b$ (with all sizes being compatible) must have a solution.

   (h) The set of complex vectors of the form $(z, \overline{z})^T$ for $z \in \mathbb{C}$ form a subspace $\mathbb{C}^2$.

   (i) Applying an elementary row operation to an orthogonal matrix gives an orthogonal matrix.

   (j) If Q is a 2×2 improper orthogonal matrix then $Q^2 = I$.

2. (16 points) Let $p(\mathbf{x}) = x^2 + 2y^2 + 4z^2 - 2xy + 4xz - 4yz - 2ax - 2bz + c$, where $\mathbf{x} = (x, y, z)$.

   (a) (3 points) Find matrix K and vector $\mathbf{f}$ such that $p(\mathbf{x}) = \mathbf{x}^T K\mathbf{x} - 2\mathbf{x}^T\mathbf{f} + c$.

   (b) (7 points) Using part (a), find conditions on a, b, c such that $p(\mathbf{x})$ has a finite minimum. Your conditions must be as general as possible.

   (c) (6 points) Let a, b, c be fixed and satisfy conditions from part (b), find the minimizer (or a set of minimizers, if it is not unique) and the minimum of $p(\mathbf{x})$.

3. (14 points) Consider the linear map $L : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$L(x, y, z) = (x + 3y + 2z, x - 4z, y + 3z)^T.$$

   (a) (6 points) Find the matrix representation of L with respect to the standard basis.

   (b) (8 points) Find the matrix representation of L with respect to the basis:

$$S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

4. (16 points) The parts of this problem are related.

   (a) (8 points) Find a symmetric matrix, A, that has eigenvalues $\lambda_1 = 1, \lambda_2 = 10, \lambda_3 = 0$ with respective eigenvectors $v_1 = (1, 2, 0), v_2 = (2, -1, 0)$, and $v_3 = (0, 0, 1)$.

   (b) (8 points) Find e^A.

5. (18 points) Let $A = \begin{pmatrix} 0 & 1 & 0 \\ -4 & 4 & 0 \\ -2 & 2 & 1 \end{pmatrix}$

   (a) (6 points) Find the eigenvalues of A and their multiplicities.

   (b) (4 points) What is J, the Jordan Canonical Form of A?

   (c) (8 points) Find the matrix S such that $A = SJS^{-1}$.

6. (16 points) Let A be the matrix with the SVD given by

$$A = \begin{pmatrix} 3/5 & 0 \\ 0 & 1 \\ 4/5 & 0 \end{pmatrix} \begin{pmatrix} 15 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1/3 & 2/3 & -2/3 \\ -2/3 & 2/3 & 1/3 \end{pmatrix}$$

   (a) (4 points) What is the rank of A? Does A have an inverse?

   (b) (6 points) Find the best rank 1 approximation for A.

   (c) (6 points) Find the pseudoinverse of A and use it to find the least squares solution to $Ax = b$ where $b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.