Write your name below. This exam is worth 100 points. On each problem (except for problem 1), you must show all your work to receive credit on that problem. You are NOT allowed to use your notes, book, calculator, or any other electronic devices.

Name:

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1. (21 points: 3 each) If the statement is **always true** mark "TRUE"; if it is possible for the statement to be false then mark "FALSE". **No justification is necessary**.

   (a) A $4 \times 4$ matrix with the only distinct eigenvalues being $-1, 0, 6$ must be diagonalizable.

   (b) Every non-trivial vector subspace has an orthonormal basis.

   (c) Let $|| \cdot ||$ be a norm coming from a real inner product $< \cdot, \cdot >$. If $||x + y||^2 = ||x||^2 + ||y||^2$ then $x$ and $y$ must be orthogonal.

   (d) Let $< \cdot, \cdot >$ be an inner-product on a complex vectors space. Then $< w, v >=< v, w >$ .

   (e) A matrix whose columns form an orthogonal basis of $\mathbb{R}^n$ is orthogonal.

   (f) Let $W \subset V$ where $V$ is a vector space with $W = span\{v_1, v_2, v_3\}$ and inner product $< \cdot, \cdot >$. The orthogonal projection of a vector $v \in V$ is always given by

   $$Pv =< v, v_1 > \frac{v_1}{||v_1||^2} + < v, v_2 > \frac{v_2}{||v_2||^2} + < v, v_3 > \frac{v_3}{||v_3||^2}.$$

   (g) Let $A$ be a square matrix with $det(A) = 0$ then zero must be an eigenvalue of $A$.

**Solutions**:

(a) **False**, since it has only three distinct eigenvalues it may not be complete, which is a requirement for diagonalization.

(b) **True**, every subspace has a basis and we can always make it orthonormal using Gram-Schmidt.

(c) **True**, expanding out the left-hand side we see that

$$||x + y||^2 = ||x||^2 + 2 < x, y > + ||y||^2$$

and to get equality we need $< x, y >= 0$, implying orthogonality.

(d) **False**, we have conjugate symmetry in complex vector spaces.

(e) **False**, the columns have to form an orthonormal basis.

(f) **False**, the formula holds if the vectors are orthogonal.

(g) **True**, we know that $det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$ where the $\lambda_i's$ are the eigenvalues.

2. (20 points) Let $C = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & -2 \\ -2 & -2 & 7 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}$.

   (a) (10 points) Find the associated quadratic form $q(x, y, z) = \mathbf{x}^T C\mathbf{x}$. Show that this quadratic form is positive definite.

   (b) (10 points) Find the Gram matrix $K$ matrix corresponding to $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ using the inner product $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T C\mathbf{y}$. Does $K$ have null directions? If yes, find them. If no, justify why. (For this part you may assume that $C > 0$).

**Solutions**:

(a) We can find the quadratic form and complete the squares to show that it is positive definite:

$$\begin{aligned} q(x, y, z) &= x^2 - 4xz + 2y^2 - 4yz + 7z^2 \\ &= (x^2 - 4xz + 4z^2) + 2(y^2 - 2yz + z^2) + z^2 \\ &= (x - 2z)^2 + 2(y - z)^2 + z^2 \\ &= q_1^2 + q_2^2 + q_3^2. \end{aligned}$$

Note that $q(x, y, z) \geq 0$ as it is a sum of non-negative numbers/square. We need to show that $q(x, y, z) > 0$ for all $\mathbf{x} \neq 0$. Equivalently, we can show that $q(x, y, z) = 0$ implies that $\mathbf{x} = 0$. $q(x, y, z) = 0$ only when each of the square terms is 0, i.e. $q_i = 0$ for $i = 1, 2, 3$. $q_3 = 0$ only when $z = 0$. $q_1 = 0$ when $x = 2z$ and $q_2 = 0$ when $y = z$, but $z = 0$, which implies that $x = y = 0$ as well.

Alternatively, you can show that $C > 0$ using any method you want, and state this implies that the quadratic form is also positive definite.

(b) Let $A$ be a matrix whose columns are $v_i$'s, i.e. $A = [\mathbf{v}_1\, \mathbf{v}_2\, \mathbf{v}_3]$. Then $K = A^T CA$. Using matrix multiplications, we get

$$K = \begin{bmatrix} 3 & -1 & 7 \\ -1 & 15 & -17 \\ 7 & -17 & 31 \end{bmatrix}.$$

In one of the previous homework problems, we showed that null directions of $K$ are non-zero vectors in the $\ker K$ (or equivalently $\ker A$). We solve the homogeneous system $K\mathbf{z} = \mathbf{0}$ (or $A\mathbf{z} = \mathbf{0}$):

$$\begin{bmatrix} 3 & -1 & 7 \\ -1 & 15 & -17 \\ 7 & -17 & 31 \end{bmatrix} \Longrightarrow \cdots \Longrightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then the null directions of $K$ are elements of $\operatorname{Span}(-2, 1, 1)^T$ excluding the zero vector.

3. (20 points)

   Let $A = \begin{pmatrix} 0 & 2 & -2 \\ 1 & 1 & 0 \\ 4 & -4 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 0 & -1 \\ 1 & 2 & -1 \\ -1 & 0 & 3 \end{pmatrix}$.

   (a) (6 points) Find the eigenvalues of $A$ and their algebraic multiplicities.

   (b) (8 points) $B$ has eigenvalue 2 with algebraic multiplicity of 2 and eigenvalue 4 with algebraic multiplicity of 1. Find all the eigenvectors of $B$ and show that it is a complete matrix.

   (c) (6 points) What are the matrices $S$ and $D$ such that $B = SDS^{-1}$? You do not need to calculate $S^{-1}$.

**Solutions**:

The characteristic polynomial of $A$ is

(a) $\det \begin{pmatrix} -\lambda & 2 & -2 \\ 1 & 1-\lambda & 0 \\ 4 & -4 & 5-\lambda \end{pmatrix} = (1-\lambda)\det \begin{pmatrix} -\lambda & -2 \\ 4 & 5-\lambda \end{pmatrix} - \det \begin{pmatrix} 2 & -2 \\ -4 & 5-\lambda \end{pmatrix}$

$= (1-\lambda)\left(\lambda^2 - 5\lambda + 8\right) - (10 - 2\lambda - 8)$

$= -\lambda^3 + 6\lambda^2 - 11\lambda + 6$

$= (1-\lambda)(2-\lambda)(3-\lambda)$

Our eigenvalues are 1, 2, and 3, each with algebraic multiplicity 1.

(b) To find the eigenvectors we need to find the kernels of the matrices:

$B - 2I = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{pmatrix}$ and $B - 4I = \begin{pmatrix} -1 & 0 & -1 \\ 1 & -2 & -1 \\ -1 & 0 & -1 \end{pmatrix}$

The REF of $B - 2I$ is $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, so it has 2 free variables and eigenvectors $v_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

The REF of $B - 4I$ is $\begin{pmatrix} -1 & 0 & -1 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{pmatrix}$, which has just one free variable, and $v_3 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$

The number of eigenvectors, and hence the geometric multiplicity of each of the eigenvalues equals the algebraic multiplicity.

Each eigenvalue is therefore complete and $B$ is a complete matrix.

(c) One possible pair of matrices is $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ and $S = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix}$

4. (20) points. Let $A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ and $K = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix}$.

   (a) (5 points) Find a basis for the image of $A$.

   (b) (9 points) Using the inner product defined by $\langle x, y \rangle = x^T K y$, find an orthogonal basis for the image of $A$. You may assume $K$ is positive definite.

   (c) (5 points) Using the same inner product, find an orthonormal basis for the image of $A$.

**Solution**:

(a) We find that since $A$ is square and $\det(A) = 2$, the image of $A$ is $\mathbb{R}^3$. We can therefore take the standard basis for $\mathbb{R}^3$ as our basis:

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

(b) These basis vectors are not orthogonal when our inner product is given above, so we must use Gram-Schmidt to make them orthogonal.

$$v_1 = e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\langle e_2, v_1 \rangle = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 2 & 0 \\ 2 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 2$$

$$||v_1||^2 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 2 & 0 \\ 2 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 2$$

$$v_2 = e_2 - \frac{\langle e_2, v_1 \rangle}{||v_1||^2} v_1 = e_2 - v_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$\langle e_3, v_1 \rangle = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 2 & 0 \\ 2 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0$$

$$\langle e_3, v_2 \rangle = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 2 & 0 \\ 2 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = 1$$

$$||v_2||^2 = \begin{pmatrix} -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 2 & 0 \\ 2 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = 1$$

$$v_3 = e_3 - \frac{\langle e_3, v_1 \rangle}{||v_1||^2} v_1 - \frac{\langle e_3, v_2 \rangle}{||v_2||^2} v_2 = e_3 - v_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

(c) To make our basis orthonormal we divide each of our orthogonal basis vectors by its norm with the K inner product. We have already found that

$||v_1||^2 = 2$

$||v_2||^2 = 1$

Now we just need to find

$$||v_3||^2 = \begin{pmatrix} 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 2 & 0 \\ 2 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 1$$

So our orthonormal basis is

$$\left\{ \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

5. (19 points) Find a symmetric orthogonal matrix $P$ whose first row is $\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$.

**Solutions:** Since $P$ is symmetric it must have the form

$$P = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & x & y \\ \frac{2}{3} & y & z \end{pmatrix}.$$

Recall that the columns of an orthogonal matrix are orthogonal and have norm one. Thus, to find $x$ and $y$ we see that columns one and two have to be orthogonal. This gives a condition that must be satisfied: $\frac{2}{9} + \frac{2}{3}x + \frac{2}{3}y = 0$ or simplified to

$$(C1)\ 1 + 3x + 3y = 0.$$

Moreover, we know that the second column must have norm one, giving a second condition: $\frac{4}{9} + x^2 + y^2 = 1$ or simplified to

$$(C2)\ 9x^2 + 9y^2 = 5.$$

We can solve $(C1)$ for $y$ in terms of $x$ and plug into $(C2)$. We see that $y = -\frac{(1+3x)}{3}$ and so

$$9x^2 + 9\left(\frac{1+3x}{3}\right)^2 = 5 \Leftrightarrow 9x^2 + 3x - 2 = 0 \Leftrightarrow (3x-1)(3x+2) = 0.$$

So there are two cases that you can consider (although you do not have to consider both)!

- Case 1: $x = 1/3$ the $y = -2/3$. Moving on, since the first and third columns are orthogonal we have the equation: $\frac{2}{9} + \frac{2}{3}y + \frac{2}{3}z = 0$ or $z = \frac{1}{3}$. This gives the matrix

$$P = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{pmatrix}.$$

- Case 2: $x = -2/3$ then $y = 1/3$. The third condition remains $\frac{2}{9} + \frac{2}{3}y + \frac{2}{3}z = 0$, but as $y = 1/3$ we see that $z = -\frac{2}{3}$. This gives the matrix

$$P = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \end{pmatrix}.$$

Either of these two matrices will work.