Write your name below. This exam is worth 100 points. On each problem (except for problem 1), you must show all your work to receive credit on that problem. You are NOT allowed to use your notes, book, calculator, or any other electronic devices.

Name:

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1. (21 points: 3 each) If the statement is **always true** mark "TRUE"; if it is possible for the statement to be false then mark "FALSE". **No justification is necessary**.

   (a) Let B be the square matrix obtained by exchanging two rows of the square matrix A and let $det(A) < det(B)$ then A is nonsingular.

   (b) In an inner-product space if $\|f\| = \|g\|$, where $\|\cdot\|$ is the norm defined from the inner product, then $f \equiv g$.

   (c) If all entries of a 5×5 matrix A are 5, then $det(A) = 5^5$.

   (d) If A and B are symmetric invertible matrices, then ABA^{-1} is also symmetric and invertible.

   (e) If f and g are elements in an inner product space satisfying $\|f\| = 2, \|g\| = 4$ and $\|f + g\| = 5$, then it is possible to find the exact value of $\langle f, g \rangle$.

   (f) If $A\mathbf{x} = \mathbf{b}$ has an infinite number of solutions then $A\mathbf{x} = \mathbf{0}$ has an infinite number number of solutions as well.

   (g) Let $v, w \in \mathbb{R}^3$ it holds that $\|w\| \leq \|v\| + \|w + v\|$, where $\|\cdot\|$ is any norm in $\mathbb{R}^3$.

Solution:

(a) **True**, note that $det(A) = -det(B)$ so if $det(A) = 0$ then $det(B) = 0$, which contradicsts the fact that $det(A) < det(B)$.

(b) **False**, consider the space $\mathbb{R}^2$ with the dot product, then $\|e_1\| = \|e_2\|$ but $e_1 = (1, 0)^T$ and $e_2 = (0, 1)^T$.

(c) **False**, the columns are linearly dependent so the determinant is zero.

(d) **False**, the product of invertible matrices is invertible, but $(ABA^{-1})^T = (A^T)^{-1}B^TA^T = A^{-1}BA \neq ABA^{-1}$. So this matrix need not be symmetric.

(e) **True**, we have that $5^2 = \|f + g\|^2 = \langle f + g, f + g \rangle = \|f\|^2 + 2\langle f, g \rangle + \|g\|^2 = 2^2 + 2\langle f, g \rangle + 4^2$. Solving for $< f, g >$ we obtain that it is equal to 5/2.

(f) **True**, the nonuniqueness of a system of linear equations comes from the homogeneous problem.

(g) **True**, note that $w = -v + v + w$ and by the triangle inequality we get the result: $\|w\| \leq \|v\| + \|v + w\|$.

2. (19 points) Consider the following matrix A

$$A = \begin{bmatrix} 0 & 3 & 0 & 2 \\ 1 & 2 & -3 & 0 \\ 2 & 5 & -4 & 3 \\ -3 & -4 & 7 & 0 \end{bmatrix}.$$

(a) (7 points) Find the permutation matrix P such that $B := PA$ is symmetric. Show both P and B.

(b) (12 points) Can B be factored as LDL^T? If yes, find the factorization. If no, justify why it cannot be factored.

Solutions:

(a) Note that the last three rows already look symmetric. Then we have

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & -3 & 0 \\ 2 & 5 & -4 & 3 \\ -3 & -4 & 7 & 0 \\ 0 & 3 & 0 & 2 \end{bmatrix}.$$

(b) Note that since B is symmetric, it could be possible. According to the theorem, we need to know if B is regular. We attempt to perform LU factorization, first:

$$B = \begin{bmatrix} 1 & 2 & -3 & 0 \\ 2 & 5 & -4 & 3 \\ -3 & -4 & 7 & 0 \\ 0 & 3 & 0 & 2 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \times & 1 & 0 & 0 \\ \times & \times & 1 & 0 \\ \times & \times & \times & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & \times & \times & \times \\ 0 & 0 & \times & \times \\ 0 & 0 & 0 & \times \end{bmatrix}$$

$$\xrightarrow[R_3=R_3+3R_1]{R_2=R_2-2R_1} B = \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & -2 & 0 \\ 0 & 3 & 0 & 2 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -3 & \times & 1 & 0 \\ 0 & \times & \times & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & \times & \times \\ 0 & 0 & 0 & \times \end{bmatrix}$$

$$\xrightarrow[R_4=R_4-2R_2]{R_3=R_3-2R_2} B = \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -6 & -6 \\ 0 & 0 & -6 & -7 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -3 & 2 & 1 & 0 \\ 0 & 3 & \times & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -6 & -6 \\ 0 & 0 & 0 & \times \end{bmatrix}$$

$$\xrightarrow{R_4=R_4-R_3} B = \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -6 & -6 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -3 & 2 & 1 & 0 \\ 0 & 3 & 1 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -6 & -6 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Hence. B is regular and can be factored as LDL^T:

$$\begin{bmatrix} 1 & 2 & -3 & 0 \\ 2 & 5 & -4 & 3 \\ -3 & -4 & 7 & 0 \\ 0 & 3 & 0 & 2 \end{bmatrix} = B = LDL^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -3 & 2 & 1 & 0 \\ 0 & 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

3. (20 points: 10 each)

   The following two problems are unrelated.

   (a) Determine if the following matrices are linearly independent

   $$\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}, \begin{pmatrix} 3 & -1 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & -5 \\ -4 & 0 \end{pmatrix}.$$

   (b) Let $V = \mathbb{R}^4$ and $W \subset V$ be the space spanned by the vectors: $\begin{pmatrix} 1 \\ -2 \\ 5 \\ -3 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 1 \\ -4 \end{pmatrix}, \begin{pmatrix} 3 \\ 8 \\ -3 \\ -5 \end{pmatrix}.$

   Find a basis and dimension for W.

Solution:

(a) To check for linear independence we solve the system:

$$a\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} + b\begin{pmatrix} 3 & -1 \\ 2 & 2 \end{pmatrix} + c\begin{pmatrix} 1 & -5 \\ -4 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Which gives the following system of equations:

$$\begin{pmatrix} 1 & 3 & 1 \\ 2 & -1 & -5 \\ 3 & 2 & -4 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = c\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Finding the REF of the matrix we get:

$$\begin{pmatrix} 1 & 3 & 1 \\ 2 & -1 & -5 \\ 3 & 2 & -4 \\ 1 & 2 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 3 & 1 \\ 0 & -7 & -7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that we have one free variable, which implies that there are an infinite number of solutions and so the matrices are linearly dependent. For example, you may take $a = 2, b = -1, c = 1$ as a nontrivial solution.

(b) (Solution 1) Let A be the 3×4 matrix whose rows are made up of these vectors, we can find the REF of A

$$A = \begin{pmatrix} 1 & -2 & 5 & -3 \\ 2 & 3 & 1 & -4 \\ 3 & 8 & -3 & -5 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Since row operations do not change the span of the row space, then

$$\text{basis for } W = \left\{(1, -2, 5, -3)^T, (0, 7, -9, 2)^T\right\}$$

and the dimension of W is two.

(Solution 2) Another solution (albeit slightly harder) is to consider the matrix whose columns are the vectors and find it REF:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -2 & 3 & 8 \\ 5 & 1 & -3 \\ -3 & -4 & -5 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 3 \\ 0 & 7 & 14 \\ 0 & -9 & -18 \\ 0 & 2 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 3 \\ 0 & 7 & 14 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Here we are looking for the column space so we see that columns one and two are pivot columns and so select the pivot columns from the original matrix A:

$$\text{basis for } W = \left\{(1, -2, 5, -3)^T, (2, 3, 1, -4)^T\right\}$$

4. (19 points)

The following two questions are unrelated.

(a) (9 points) Let $V = \mathbb{R}^3$ and $W = \left\{(x, y, z)^T \in V : \; x^2 - 2xy + y^2 - z^2 = 0\right\}$. Is W a vector subspace of V? Prove or disprove.

(b) (10 points) Consider $\mathcal{F}(I)$, the vector space of real valued functions on an interval I. Do the solutions to the differential equation

$$y'' + 5y' + 2y = 0$$

form a subspace of $\mathcal{F}(I)$? Prove that they do or show that they do not.

Solution:

(a) W is not a vector space as it is not closed under vector addition:

Let $w_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and $w_2 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ be members of W, then

$w_1 + w_2 = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}$, which is not a member of W.

(b) This is a subspace. It is non-empty ($y = 0$ is a solution) and closed under both scalar multiplication and vector addition:

For any $c \in \mathbb{R}$ we have
$(cy)'' + 5(cy)' + 2(cy) = c(y)'' + 5c(y)' + 2cy$

$= c(y'' + 5y' + 2y) = c(0) = 0.$

For any 2 solutions to the differential equation we have
$(y_1 + y_2)'' + 5(y_1 + y_2)' + 2(y_1 + y_2) = y_1'' + y_2'' + 5y_1' + 5y_2' + 2y_1 + 2y_2$

$= (y_1'' + 5y_1' + 2y_1) + (y_2'' + 5y_2' + 2y_2) = 0$

5. (21 points)

Let $A = \begin{pmatrix} 1 & 2 & 1 & 2 & 0 \\ 2 & 4 & 3 & 4 & 1 \\ 1 & 2 & 2 & 2 & 1 \end{pmatrix}$.

(a) (3 points) What is the rank of A?

(b) (3 points) What is dim coker A?

(c) (5 points) Find a basis for the image of A.

(d) (5 points) Find a basis for the coimage of A.

(e) (5 points) Find a basis for the kernel of A.

Solution:

To answer these questions, we need to first calculate the REF of A:

$$A = \begin{pmatrix} 1 & 2 & 1 & 2 & 0 \\ 2 & 4 & 3 & 4 & 1 \\ 1 & 2 & 2 & 2 & 1 \end{pmatrix}$$

$$\xrightarrow[R_3=R_3-R_1]{R_2=R_2-2R_1} \begin{pmatrix} 1 & 2 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{R_3=R_3-R_2} \begin{pmatrix} 1 & 2 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = U$$

(a) A has 2 pivot columns, so rank $A = 2$

(b) A has 3 rows and rank $A = 2$ so the dimension of the coker is $3 - 2 = 1$.

(c) The first and third columns of A are its pivot columns, so we take them as our basis for the image: $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$.

(d) The REF has 2 non-zero rows. Their transposes form a basis for the coimage: $\begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$

(e) A has 5 columns and is rank 2, so the kernel will have $5 - 2 = 3$ basis vectors. From the REF we see that x_2, x_4, and x_5 are free variables.

Solving for the three vectors using $Ux = 0$ gives: $\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$ for our basis vectors.