Write your name and your professor's name or your section number in the top right corner of your paper. You are allowed to use textbooks and notes, but you may not ask anyone for help except the professors. To receive full credit on a problem you must show **sufficient justification for your conclusion** unless explicitly stated otherwise. You may use a calculator (or software) to check your answers, but you may not submit decimal answers: you have to show your work.

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1. (20 points: 2 each) If the statement is **always true** mark "TRUE"; if it is possible for the statement to be false then mark "FALSE." **No justification is necessary.**

   T(a) If $\mathbf{A}$ is skew-symmetric and invertible then $\mathbf{A}^{-1}$ is also skew-symmetric.

   T(b) If $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ form a basis for $\mathbb{R}^n$ then they also form a basis for $\mathbb{C}^n$.

   T(c) Given a basis of $\mathbb{R}^n$, it is possible to find an inner product such that the basis is orthogonal with respect to the inner product.

   Solution: Let the vectors be columns of $\mathbf{A}$, and $\mathbf{A} = \mathbf{PSQ}^T$ be the SVD of $\mathbf{A}$. Then the SPD matrix $\mathbf{K} = \mathbf{PS}^{-2}\mathbf{P}^T$ makes the vectors orthogonal.

   F(d) If $\boldsymbol{x}$ is a least-squares solution to $\mathbf{Ax} = \boldsymbol{b}$, then the residual $\mathbf{b} - \mathbf{Ax}$ is orthogonal to $\boldsymbol{b}$. (You may assume orthogonality is defined using the dot product, and the least-squares problem is unweighted.)

   Solution: $\mathbf{Ax}$ is the orthogonal projection of $\mathbf{b}$ into the range of $\mathbf{A}$, so the residual is orthogonal to the range of $\mathbf{A}$ (not to $\mathbf{b}$) according to the definition of orthogonal projection.

   F(e) If $\lambda_1, \boldsymbol{v}_1$ and $\lambda_2, \boldsymbol{v}_2$ are both eigenvalue/eigenvector pairs for $\mathbf{A}$, then $\lambda_1\boldsymbol{v}_1 + \lambda_2\boldsymbol{v}_2$ is an eigenvector of $\mathbf{A}$.

   T(f) If a square matrix $\mathbf{A}$ has the property that $|a_{ii}| > \sum_{j\neq i}|a_{ij}|$ for every i, then $\mathbf{A}$ is invertible.

   T(g) Suppose that $\mathbf{A}$ is diagonalizable with $\mathbf{A} = \mathbf{S\Lambda S}^{-1}$, and that $\boldsymbol{y}^T$ is a row of $\mathbf{S}^{-1}$. True or False: $\mathbf{A}^T\boldsymbol{y} = \lambda\boldsymbol{y}$ for some eigenvalue λ.

   F(h) Suppose that $\mathbf{A}$ is a square matrix with real entries and an orthogonal eigenvector basis. True or False: $\mathbf{A}$ is symmetric.

   T(i) Suppose that $\mathbf{Z}$ is an orthogonal matrix and $\mathbf{A} = \mathbf{ZB}$ where $\mathbf{A}$ and $\mathbf{B}$ are both real. True or false: $\mathbf{A}$ and $\mathbf{B}$ have the same singular values.

   T(j) Suppose that $\mathbf{A}$ is 4×10 with rank 4, and that $\mathbf{A}^+$ is the pseudoinverse of $\mathbf{A}$. True or false: $\mathbf{AA}^+ = \mathbf{I}$.

2. (16 points) Find the Cholesky decomposition of the following matrix $\mathbf{A} = \begin{bmatrix} 9 & -9 & 9 \\ -9 & 10 & -7 \\ 9 & -7 & 14 \end{bmatrix}$.

   Solution: First find the REF:
$$\begin{bmatrix} 9 & -9 & 9 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$
   Then divide each row by the square root of the diagonal element, and take the transpose:
$$\begin{bmatrix} 3 & 0 & 0 \\ -3 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

3. (20 points) Let

$$L(x,y,z) = \begin{pmatrix} 2(y-z)+x \\ 4z+2y \\ -z \end{pmatrix}.$$

Find the matrix representation of L with respect to the following basis of $\mathbb{R}^3$:
$\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \right\}$.

Solution: First find the matrix representation with respect to the standard basis by plugging in the standard basis vectors and placing the output as columns of

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -2 \\ 0 & 2 & 4 \\ 0 & 0 & -1 \end{bmatrix}.$$

Next arrange the new basis vectors as columns of a matrix

$$\mathbf{S} = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The solution is found using $\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \begin{bmatrix} -1 & 6 & 2 \\ -1 & 4 & 2 \\ 0 & 0 & -1 \end{bmatrix}$. Although you don't actually need $\mathbf{S}^{-1}$ to find the solution, here it is for reference, along with $\mathbf{AS}$:

$$\mathbf{S}^{-1} = \begin{pmatrix} 1 & 0 & -2 \\ \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{AS} = \begin{pmatrix} -1 & 6 & 0 \\ -2 & 6 & 4 \\ 0 & 0 & -1 \end{pmatrix}$$

4. (20 points) Suppose that you have the following data from 100 people: weight w_i, height h_i, age t_i and blood pressure p_i (where $i = 1, \ldots, 100$). You decide to model the person's blood pressure p as a function of the other factors as follows $p = \beta_0 + \beta_1 \frac{w}{h^2} + \beta_2 t + \beta_3 t^2$. What are the entries of the matrix $\mathbf{A}$ and vector $\vec{b}$ such that the least-squares solution of $\mathbf{A}\vec{x} = \vec{b}$ is the vector of linear regression coefficients $\beta_0, \ldots, \beta_3$?

The linear system is obtained by plugging the data into the model, which yields the system

$$\begin{aligned} \beta_0 + \beta_1 \frac{w_1}{h_1^2} + \beta_2 t_1 + \beta_3 t_1^2 &= p_1 \\ &\vdots \\ \beta_0 + \beta_1 \frac{w_{100}}{h_{100}^2} + \beta_2 t_{100} + \beta_3 t_{100}^2 &= p_{100} \end{aligned}$$

This linear system can be written in matrix form $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ with

$$\mathbf{A} = \begin{bmatrix} 1 & \frac{w_1}{h_1^2} & t_1 & t_1^2 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \frac{w_{100}}{h_{100}^2} & t_{100} & t_{100}^2 \end{bmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} p_1 \\ \vdots \\ p_{100} \end{pmatrix}.$$

5. (24 points) Find the singular value decomposition of $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$.

Solution: The Gram matrix is

$$\mathbf{A}^T\mathbf{A} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}.$$

The eigenvalues of the Gram matrix are 0 and 2, so there is only one singular value $\sigma_1 = \sqrt{2}$. The associated singular vector is $(1,1)^T$, which has to be normalized by dividing by $\sqrt{2}$. There is only one $\boldsymbol{p}_i$ vector, found using $\boldsymbol{p}_1 = \sigma_1^{-1}\mathbf{A}\boldsymbol{q}_1$. The final answer is

$$\mathbf{P} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}, \ \mathbf{\Sigma} = 2, \ \ \mathbf{Q} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$