Write your name and your professor's name or your section number in the top right corner of your paper. You are allowed to use textbooks and notes, but you may not ask anyone for help except the proctors. To receive full credit on a problem you must show **sufficient justification for your conclusion** unless explicitly stated otherwise.

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1. (30 points: 3 each) If the statement is **always true** mark "TRUE"; if it is possible for the statement to be false then mark "FALSE." **No justification is necessary.**

   F(a) If $L(\boldsymbol{x}) = \mathbf{A}\mathbf{x}$ is a linear function from $\mathbf{R}^n$ to $\mathbf{R}^n$, and the columns of $\mathbf{B}$ form a basis of $\mathbb{R}^n$, then $\mathbf{B}\mathbf{A}\mathbf{B}^{-1}$ is the matrix representation of L with respect to the new basis.

   Solution: The correct answer is $\mathbf{B}^{-1}\mathbf{A}\mathbf{B}$.

   F(b) If $\|\cdot\|$ is a norm on vector space V, then the formula $\frac{1}{4}\left(\|\boldsymbol{u}+\boldsymbol{v}\|^2 - \|\boldsymbol{u}-\boldsymbol{v}\|^2\right)$ defines an inner product $\langle \boldsymbol{u}, \boldsymbol{v} \rangle$ on V.

   Solution: This expression only defines an inner product when the norm comes from an inner product. Not every norm comes from an inner product.

   F(c) The 1-norm of an $n \times n$ matrix $\mathbf{A}$ is $\max_i \sum_{j=1}^n |a_{ij}|$.

   Solution: The 1-norm of an $n \times n$ matrix $\mathbf{A}$ is $\max_j \sum_{i=1}^n |a_{ij}|$.

   T(d) Suppose $\mathbf{A}$ has full column rank, and QR factorization $\mathbf{QR}$ (the Gram-Schmidt version). True or False: $\mathbf{R}^T$ is the Cholesky factor of the Gram matrix $\mathbf{A}^T\mathbf{A}$.

   T(e) The determinant of a Householder elementary reflection matrix is -1.

   T(f) If c_i are the coordinates of $\boldsymbol{b}$ with respect to an orthonormal basis, then $\|\boldsymbol{b}\|^2 = \sum_i c_i^2$ (the norm is the one associated with the inner product).

   T(g) Let V be a finite-dimensional inner product space and let W be a subspace of V. True or False: $\left(W^\perp\right)^\perp = W$.

   F(h) The orthogonal projection of $\boldsymbol{b}$ onto the range of $\mathbf{A}$ (using the dot product) is given by $\mathbf{A}\left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\boldsymbol{b}$.

   Solution: The formula only works if $\mathbf{A}$ has full column rank. If $\mathbf{A}$ does not have full column rank then $\mathbf{A}^T\mathbf{A}$ is not invertible and the formula does not apply.

   T(i) The orthogonal complement of the kernel (also called the null space) of $\mathbf{A}$ is the corange (also called the row space or co-image) of $\mathbf{A}$. (Orthogonality is defined using the dot product.)

   T(j) If $\boldsymbol{b}$ is orthogonal to cokernel (also called the left-hand null space) of $\mathbf{A}$, then $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ has a solution.

2. (15 points) Suppose that $\mathbf{A}\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}-1\\2\\0\end{pmatrix}$ and $\mathbf{A}\begin{pmatrix}1\\-1\\2\end{pmatrix} = \begin{pmatrix}0\\1\\3\end{pmatrix}$. Solve $\mathbf{A}\mathbf{x} = \begin{pmatrix}1\\0\\6\end{pmatrix}$.

   Solution: First solve
$$c_1\begin{pmatrix}-1\\2\\0\end{pmatrix} + c_2\begin{pmatrix}0\\1\\3\end{pmatrix} = \begin{pmatrix}1\\0\\6\end{pmatrix}.$$
   The solution can be obtained by looking at the first and last equations, and is $c_1 = -1$ $c_2 = 2$. Next find the solution $\boldsymbol{x}$ as
$$c_1\begin{pmatrix}1\\0\\1\end{pmatrix} + c_2\begin{pmatrix}1\\-1\\2\end{pmatrix} = \begin{pmatrix}1\\-2\\3\end{pmatrix} = \boldsymbol{x}.$$

3. (20 points) Prove that the following expression defines an inner product:
$$\langle \boldsymbol{u}, \boldsymbol{v}\rangle = 2u_1(v_1 - 2v_2) + u_2(-4v_1 + 9v_2 + v_3) + u_3(v_2 + 4v_3)$$

   Solution: We will show that $\langle \boldsymbol{u}, \boldsymbol{v}\rangle = \boldsymbol{u}^T\mathbf{K}\boldsymbol{v}$ where $\mathbf{K}$ is a symmetric positive definite (SPD) matrix. This proves that the formula defines an inner product.

   First, note that
$$\mathbf{K} = \begin{bmatrix}2 & -4 & 0\\-4 & 9 & 1\\0 & 1 & 4\end{bmatrix}.$$
   Clearly this is symmetric. Row reducing never requires a row permutation, and leads to the REF
$$\begin{bmatrix}2 & -4 & 0\\0 & 1 & 1\\0 & 0 & 3\end{bmatrix}.$$
   The diagonal entries are all positive, so the matrix is SPD.

4. (20 points) Find the QR factorization of $\mathbf{A} = \begin{bmatrix} 1 & -2 & -1 \\ 1 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ using the Gram-Schmidt version of QR.

Solution:

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix} - \frac{-4}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\boldsymbol{v}_3 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} - \frac{0}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{1}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

Normalize the columns, then arrange them as columns of $\mathbf{Q}$:

$$\mathbf{Q} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix}$$

Obtain $\mathbf{R}$ using $\mathbf{Q}^T\mathbf{A}$:

$$\mathbf{R} = \begin{bmatrix} \sqrt{2} & -2\sqrt{2} & 0 \\ 0 & 1 & 1 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$$

5. (15 points) Find the critical point of this function, and explain whether it is a maximizer, minimizer, or neither:

$$q(x, y, z) = 2x^2 - 8xy + 9y^2 + 2yz + 4z^2 + 4x - 12y - 10z - \sqrt{581}$$

Solution: The gradient is

$$\nabla q = \begin{pmatrix} 4x - 8y + 4 \\ -8x + 18y + 2z - 12 \\ 2y + 8z - 10 \end{pmatrix}$$

Setting this to zero yields a linear system of equations for the critical point. The augmented matrix for this linear system is

$$\left[\begin{array}{ccc|c} 4 & -8 & 0 & -4 \\ -8 & 18 & 2 & 12 \\ 0 & 2 & 8 & 10 \end{array}\right]$$

The row echelon form is obtained without pivoting, and has the form

$$\left[\begin{array}{ccc|c} 4 & -8 & 0 & -4 \\ 0 & 2 & 2 & 4 \\ 0 & 0 & 6 & 6 \end{array}\right]$$

The facts that the REF is obtained without needing to pivot, and that the diagonal elements are all positive imply that the Hessian is positive definite, so that the critical point is a minimizer of the function. The solution (the minimizer) is

$$\boldsymbol{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$