Write your name and your professor's name or your section number in the top right corner of your paper. You are allowed to use textbooks and notes, but you may not ask anyone for help except the proctors. To receive full credit on a problem you must show **sufficient justification for your conclusion** unless explicitly stated otherwise.

---

1. (30 points: 3 each) If the statement is **always true** mark "TRUE"; if it is possible for the statement to be false then mark "FALSE." **No justification is necessary.**

   F(a) If $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are all square invertible matrices of the same size, then $(\mathbf{B}\mathbf{C}^T\mathbf{A}^{-1})^{-1} = \mathbf{B}^{-1}(\mathbf{C}^T)^{-1}\mathbf{A}$.

   Solution: The correct expression is $\mathbf{A}\mathbf{C}^{-T}\mathbf{B}^{-1}$.

   T(b) If $\mathbf{A}$ is a square invertible (aka nonsingular) matrix, then so is $\mathbf{A}^n$ for any positive integer n.

   F(c) Let $\vec{x} = (1,2)^T$ be a column vector and $\vec{y} = (-1,2)$ be a row vector. True or false: $\det(\vec{x}\vec{y}) = 3$.

   Solution: The determinant is zero.

   T(d) If $\mathbf{A}\vec{x} = \vec{b}$ has two solutions, then it has infinitely many solutions.

   F(e) If an upper-triangular matrix is invertible, then its inverse is lower triangular.

   Solution: The inverse of an upper-triangular matrix is also upper-triangular.

   F(f) If $\mathbf{A}$ is a square matrix with $\det(\mathbf{A}) = 0$, then $\mathbf{A}\vec{x} = \vec{b}$ has no solutions.

   Solution: If $\vec{b} = \vec{0}$, then a solution exists. So since the statement is not *always* true we mark it False.

   F(g) If the rank of $\mathbf{A}$ is less than the number of columns of $\mathbf{A}$, then the linear system of equations $\mathbf{A}\vec{x} = \vec{b}$ has an infinite number of solutions.

   Solution: It is possible to construct an example where no solution exists.

   F(h) Let V be the set of 3×4 matrices with the usual definitions of addition and scalar multiplication. This is a vector space. Consider the subset W of matrices with rank 1. True or False: W is a subspace of V.

   Solution: If you multiply a rank-one matrix by 0, then it will be the zero matrix, which has rank 0 and is thus not in the set. W is not closed under scalar multiplication, so it can't be a subspace.

   T(i) If $\mathbf{A}$ is 3×4 with rank 3, then the columns of $\mathbf{A}$ span $\mathbb{R}^3$.

   F(j) If $\mathbf{A}$ is 4×6 with rank 2, then the kernel of $\mathbf{A}$ is two dimensional.

   Solution: The fundamental theorem tells us that the dimension of the kernel is the number of columns minus the rank, so $6 - 2 = 4$, not 2.

2. Let $A = \begin{bmatrix} 0 & 2 & 2 \\ 1 & 0 & 3 \\ 0 & 1 & -1 \end{bmatrix}$.

(a) (8 points) Find the permuted LU factorization of $\mathbf{A}$.

Solution A:

$$\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & -2 \end{bmatrix}$$

Solution B:

$$\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 4 \end{bmatrix}$$

You must show your work to get credit.

(b) (12 points) Use the permuted LU factorization to solve $\mathbf{A}\vec{x} = \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix}$.

Solution A:

(o) $\mathbf{P}\vec{b} = \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix}$

(i) Solve $\mathbf{L}\vec{y} = \mathbf{P}\vec{b}$, $\vec{y} = \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix}$.

(ii) Solve $\mathbf{U}\vec{x} = \vec{y}$, $\vec{x} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$.

Solution B:

(o) $\mathbf{P}\vec{b} = \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix}$

(i) Solve $\mathbf{L}\vec{y} = \mathbf{P}\vec{b}$, $\vec{y} = \begin{pmatrix} -2 \\ 2 \\ -4 \end{pmatrix}$.

(ii) Solve $\mathbf{U}\vec{x} = \vec{y}$, $\vec{x} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$.

(c) (5 points) Use the permuted LU factorization to find the determinant of $\mathbf{A}$.

Solution A: $(-1)^1(1)(2)(-2) = 4$
Solution B: $(-1)^2(1)(1)(4) = 4$.

3. (10 points) Let $V = \mathbb{R}^3$ and let W be the subset consisting of vectors $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ such that $x^2 + 2xy + y^2 = 0$. Prove that W is a subspace, or find an example showing that it is not closed under addition or scalar multiplication.

   Solution: First note that $x^2 + 2xy + y^2 = (x+y)^2 = 0$ implies that $x = -y$. Next prove that the set is closed under addition and scalar multiplication.

   (i) Let $x_1 = -y_1$ and $x_2 = -y_2$ and let $x_3 = x_1 + x_2$, $y_3 = y_1 + y_2$, $z_3 = z_1 + z_2$. In order for the set to be closed under addition we need $x_3 = -y_3$. Plugging in the definition we find

   $$\begin{aligned} x_3 &= x_1 + x_2 \\ &= -y_1 - y_2 \\ &= -(y_1 + y_2) \\ &= -y_3 \end{aligned}$$

   (ii) Let $x_1 = -y_1$ and let $x_2 = cx_1$, $y_2 = cy_1$, and $z_2 = cz_1$. In order for the set to be closed under scalar multiplication we need $x_2 = -y_2$ for every scalar c. Plugging in the definition we find

   $$\begin{aligned} x_2 &= cx_1 \\ &= c(-y_1) \\ &= -cy_1 \\ &= -y_2 \end{aligned}$$

4. (15 points) Do the following functions span the vector space of polynomials of degree ≤ 2?

   $$\left\{1, 1-x, 1+2x-x^2, x^3\right\}$$

   Solution: We begin by rephrasing the question as "Can any polynomial of degree ≤ 2 be written as a linear combination of the functions in the list?" In mathematical notation we can write this as

   $$c_1 + c_2(1-x) + c_3(1+2x-x^2) + c_4x^3 = a + bx + cx^2.$$

   The question is whether you can always find a coefficients c_1, c_2, c_3, and c_4 (that depend on a, b, and c) to make this equation true, or whether it's instead possible to choose a, b, and c in such a way that no solution exists. We can write this as a linear system

   $$\begin{array}{lrcl} 1: & c_1 + c_2 + c_3 & = & a \\ x: & -c_2 + 2c_3 & = & b \\ x^2: & -c_3 & = & c \\ x^3: & c_4 & = & 0 \end{array}$$

   This linear system is already in row echelon form, in fact the augmented matrix has the form

   $$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & a \\ 0 & -1 & 2 & 0 & b \\ 0 & 0 & -1 & 0 & c \\ 0 & 0 & 0 & 1 & 0 \end{array}\right].$$

The coefficient matrix is invertible (4×4 with rank 4), so a solution always exists.

5. Let $\mathbf{B} = \begin{bmatrix} 1 & 0 & 3 \\ -1 & 1 & -1 \\ 2 & -3 & 0 \end{bmatrix}$. The row echelon form of $\mathbf{B}$ is $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.

   (a) (4 points) Find a basis for the range (aka image) of $\mathbf{B}$.

   Solution: The first two columns of $\mathbf{B}$.

   (b) (4 points) Find a basis for the corange (aka coimage) of $\mathbf{B}$.

   Solution: The first to rows of the REF of $\mathbf{B}$.

   (c) (4 points) Find a basis for the kernel of $\mathbf{B}$.

   Solution: The homogeneous solution is

$$\begin{aligned} x &= -3z \\ y &= -2z \\ z &= z \end{aligned}$$

   so there is a single vector in the basis: $\begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix}$.

   (d) (8 points) Find a basis for the cokernel of $\mathbf{B}$.

   Solution: The REF of $\mathbf{B}^T$ is

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

   The solution to $\mathbf{B}^T \vec{x} = \vec{0}$ is

$$\begin{aligned} x &= z \\ y &= 3z \\ z &= z \end{aligned}$$

   so there is a single vector in the basis: $\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$.