Write your name below. This exam is worth 100 points. On each problem, you must show all your work to receive credit on that problem. You are allowed to use two pages of notes (one piece of paper, front and back). You are not allowed to use a calculator or any computational software.

Name: Section: (Chi/9:05/001; Grooms/11:15/002; Grooms/1:25/003)

1. (28 points: 4 each) If the statement is **always true** mark "TRUE"; if it is possible for the statement to be false then mark "FALSE". **No justification is necessary**.

_____(a) The polynomials $1 - x + 3x^2, x + x^2, x^2$ span the space of polynomials of degree at most 2.

Solution: True. Any polynomial of degree at most 2 can be written

$$p(x) = a + bx + cx^2.$$

The question is asking whether it is possible to find constants y_1, y_2, and y_3 so that

$$y_1(1 - x + 3x^2) + y_2(x + x^2) + y_3x^2 = a + bx + cx^2$$

for all possible combinations of a, b, and c. Matching powers of x we find

$$\begin{array}{llllll} x^0: & y_1 & & & = & a \\ x^1: & -y_1 & +y_2 & & = & b \\ x^2: & 3y_1 & +y_2 & +y_3 & = & c \end{array}$$

A solution exists for any right hand side, so the given functions span the space.

_____(b) The columns of a matrix $\mathbf{A}$ are linearly independent if and only if the only solution to the homogeneous linear system $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ is the trivial one $\boldsymbol{x} = \mathbf{0}$.

Solution: True. Theorem 2.21(b).

_____(c) Let $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_N$ be vectors in a vector space V, and suppose that the only way to set $\sum_{n=1}^N x_n\boldsymbol{v}_n = \mathbf{0}$ is for all the values x_n to be equal. True or false: The dimension of the span of $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_N$ is $N - 1$.

Solution: True. The vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{N-1}$ are linearly independent and still span the whole subspace, so they are a basis for the subspace, and so the dimension of the subspace must be $N - 1$.

- Proof that the vectors are linearly independent, by reductio ad absurdum (aka contradiction). Suppose that the vectors are linearly dependent, i.e. there are x_n not all zero such that

  $$x_1\boldsymbol{v}_1 + \cdots + x_{N-1}\boldsymbol{v}_{N-1} = \mathbf{0}.$$

  This implies that you can set $x_N = 0$ and then have

  $$x_1\boldsymbol{v}_1 + \cdots + x_N\boldsymbol{v}_N = \mathbf{0}.$$

  This contradicts the assumption of the problem though, because it has $\sum_{n=1}^N x_n\boldsymbol{v}_n = \mathbf{0}$ but the x_n are not all equal.

- Proof that the vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{N-1}$ still span the whole subspace. Start by noting that

$$\sum_{n=1}^{N} \boldsymbol{v}_n = \mathbf{0} \Rightarrow \boldsymbol{v}_N = -\sum_{n=1}^{N-1} \boldsymbol{v}_n.$$

We know that the full set of vectors spans the space, i.e. for any $\boldsymbol{v}$ in the span of the full set of vectors we have

$$\sum_{n=1}^{N} x_n \boldsymbol{v}_n = \sum_{n=1}^{N-1} x_n \boldsymbol{v}_n + x_N \boldsymbol{v}_N = \boldsymbol{v}.$$

Replace $\boldsymbol{v}_N$ by a linear combination of the other vectors to find

$$\sum_{n=1}^{N-1} x_n \boldsymbol{v}_n - x_N \sum_{n=1}^{N-1} \boldsymbol{v}_n = \sum_{n=1}^{N-1} (x_n - x_N) \boldsymbol{v}_n = \boldsymbol{v}.$$

So every vector in the span can be written as a linear combination of the vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{N-1}$.

_____(d) The function $F : \mathbb{R}^3 \to \mathbb{R}$ below

$$F(x, y, z) = x - y - z$$

is linear.

Solution: **True**. Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ and let $c \in \mathbb{R}$. Then

$$\begin{aligned} F(c\,\mathbf{u} + \mathbf{v}) &= (c\,u_1 + v_1) - (c\,u_2 + v_2) - (c\,u_3 + v_3) \\ &= c\,(u_1 - u_2 - u_3) + (v_1 - v_2 - v_3) \\ &= c\,F(\mathbf{u}) + F(\mathbf{v}). \end{aligned}$$

_____(e) The following function $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$

$$\langle \mathbf{v}, \mathbf{w} \rangle = v_1 w_1 - v_1 w_2 - v_2 w_1 + 4 v_2 w_2$$

is *not* an inner product.

Solution: **False**. It satisifies positivity, symmetry, and bilinearity so it is an inner product.

_____(f) The matrix $\mathbf{K} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ is positive definite.

Solution: **False**. It is positive semi-definite since for $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, we have

$$\mathbf{x}^T \mathbf{K} \mathbf{x} = x_1^2 - 2 x_1 x_2 + x_2^2 \geq 0.$$

However, the second column is -1 times the first. So it is not invertible. So it is not positive definite.

_____(g) The set of positive definite $n \times n$ matrices is a subspace of the set of $n \times n$ matrices.

Solution: False. If you multiply a positive definite matrix by -1 the result is no longer positive definite, so the set is not closed under scalar multiplication and is therefore not a subspace.

2. (32 points, 8 each) Consider the following matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 4 & 0 \\ 2 & 2 & 0 \\ 0 & 2 & 0 \end{pmatrix}$$

(a) What is a basis for range($\mathbf{A}$)? What is its dimension? Explain your answer.

(b) What is a basis for corange($\mathbf{A}$)? What is its dimension? Explain your answer.

(c) What is a basis for kernel($\mathbf{A}$)? What is its dimension? Explain your answer.

(d) What is the dimension of cokernel($\mathbf{A}$)? Explain your answer.

Solution: To find bases for these spaces, we begin by row reducing $\mathbf{A}$. Eliminating below the first pivot: -2(R1) + (R2)

$$\mathbf{A} \to \begin{pmatrix} 1 & 4 & 0 \\ 0 & -6 & 0 \\ 0 & 2 & 0 \end{pmatrix}$$

Eliminating below the second pivot:

$$\mathbf{A} \to \begin{pmatrix} \mathbf{1} & 4 & 0 \\ 0 & \mathbf{-6} & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathbf{U}$$

(a) Since the pivots are in the first two columns, the first two columns of $\mathbf{A}$ are a basis for range($\mathbf{A}$) so a basis for range($\mathbf{A}$) is $\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix} \right\}$. The dimension is 2 since there are two pivots and the dimension is equal to the rank of $\mathbf{A}$ (and there are two vectors in the basis for the space).

(b) Since the pivots are in the first two rows, the first two rows of $\mathbf{U}$ form a basis for corange($\mathbf{A}$) so a basis for corange($\mathbf{A}$) is $\left\{ \begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -6 \\ 0 \end{pmatrix} \right\}$. The dimension is 2 since there are two pivots and the dimension is equal to the rank of $\mathbf{A}$ (and there are two vectors in the basis for the space).

(c) To get a basis for kernel($\mathbf{A}$), we solve the homogeneous system $\mathbf{Az} = \mathbf{0}$ and find that solutions to the homogenous sytem have the form $z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, where $z \in \mathbb{R}$. So a basis for kernel($\mathbf{A}$) is $\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$. The dimension of the kernel is (number columns in $\mathbf{A}$ - rank($\mathbf{A}$)) = 3 - 2 = 1 (and there is one vector in the basis for the space).

(d) The dimension for the cokernel is the (number of rows in $\mathbf{A}$ - rank($\mathbf{A}$)) = 3 - 2 = 1.

3. (12 points, 6 each) Find the matrix form of the linear transformation

$$L(x, y) = \begin{pmatrix} x - 4y \\ -2x + 3y \end{pmatrix}$$

(a) with respect to the standard basis of $\mathbb{R}^2$: $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and

(b) with respect to the basis of $\mathbb{R}^2$: $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

Solution:

(a) We apply L to each of the basis vectors and get

$$\begin{aligned} L(\mathbf{e}_1) &= \begin{pmatrix} 1 \\ -2 \end{pmatrix} = 1 \cdot \mathbf{e}_1 - 2\mathbf{e}_2 \\ L(\mathbf{e}_2) &= \begin{pmatrix} -4 \\ 3 \end{pmatrix} = -4 \cdot \mathbf{e}_1 + 3 \cdot \mathbf{e}_2. \end{aligned}$$

So the matrix for L in the standard basis is $\begin{pmatrix} 1 & -4 \\ -2 & 3 \end{pmatrix}$.

(b) We apply L to each of the basis vectors and get

$$\begin{aligned} L(\mathbf{v}_1) &= \begin{pmatrix} 2 - 4 \\ -4 + 3 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix} = -1 \cdot \mathbf{v}_1 \\ L(\mathbf{v}_2) &= \begin{pmatrix} -1 - 4 \\ 2 + 3 \end{pmatrix} = \begin{pmatrix} -5 \\ 5 \end{pmatrix} = 5 \cdot \mathbf{v}_2. \end{aligned}$$

So the matrix for L in this basis is $\begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}$.

4. (28 points, 7 each points) Let $\boldsymbol{v} = \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$ and $\boldsymbol{w} = \begin{pmatrix} -1 \\ 9 \\ 0 \end{pmatrix}$.

   (a) Find all vectors that are orthogonal to both $\boldsymbol{v}$ and $\boldsymbol{w}$ when orthogonality is defined with respect to the dot product.

   (b) Find all vectors that are orthogonal to both $\boldsymbol{v}$ and $\boldsymbol{w}$ when orthogonality is defined with respect to the inner product

$$\langle \boldsymbol{a}, \boldsymbol{b} \rangle = \boldsymbol{a}^T \begin{bmatrix} 9 & 1 & 3 \\ 1 & 1 & 0 \\ 3 & 0 & 4 \end{bmatrix} \boldsymbol{b}$$

   (c) Find the Gram matrix formed from $\boldsymbol{v}$ and $\boldsymbol{w}$ using the inner product from part (b).

   (d) Is the Gram matrix from part (c) positive definite? Explain why or why not.

Solution:

(a) Let $\boldsymbol{x} = (x, y, z)^T$. $\boldsymbol{x} \cdot \boldsymbol{v} = 0$ and $\boldsymbol{x} \cdot \boldsymbol{w} = 0$ form two equations:

$$\begin{array}{rrrcl} -x & & +3z & = & 0 \\ -x & +9y & & = & 0 \end{array}$$

Row reducing produces

$$\begin{array}{rrrcl} -x & & +3z & = & 0 \\ 0 & 9y & -3z & = & 0 \end{array}$$

x and y are basic variables and z is free. Moving the free variable to the right hand side and solving produces

$$\begin{aligned} x &= 3z \\ y &= \frac{1}{3}z \\ z &= z \text{ is free.} \end{aligned}$$

This is acceptable, as is $\boldsymbol{x} = t(9, 1, 3)^T$ for any t.

(b) Let $\boldsymbol{x} = (x, y, z)^T$. Form two equations from $\langle \boldsymbol{x}, \boldsymbol{v} \rangle = 0$ and $\langle \boldsymbol{x}, \boldsymbol{w} \rangle = 0$:

$$\begin{array}{rrcl} -y & +9z & = & 0 \\ 8y & -3z & = & 0 \end{array}$$

Row reducing produces

$$\begin{array}{rrcl} -y & +9z & = & 0 \\ & 69z & = & 0 \end{array}$$

y and z are basic variables and x is free. The solution is $x =$ free, $y = z = 0$. This is acceptable, as is $\boldsymbol{x} = t(1, 0, 0)^T$ for any t.

(c)

$$k_{11} = \langle \boldsymbol{v}, \boldsymbol{v} \rangle = \boldsymbol{v}^T \begin{bmatrix} 9 & 1 & 3 \\ 1 & 1 & 0 \\ 3 & 0 & 4 \end{bmatrix} \boldsymbol{v} = \boldsymbol{v}^T \begin{pmatrix} 0 \\ -1 \\ 9 \end{pmatrix} = 27$$

$$k_{12} = k_{21} = \langle \boldsymbol{v}, \boldsymbol{w} \rangle = \boldsymbol{v}^T \begin{bmatrix} 9 & 1 & 3 \\ 1 & 1 & 0 \\ 3 & 0 & 4 \end{bmatrix} \boldsymbol{w} = \boldsymbol{v}^T \begin{pmatrix} 0 \\ 8 \\ -3 \end{pmatrix} = -9$$

$$k_{22} = \langle \boldsymbol{w}, \boldsymbol{w} \rangle = \boldsymbol{w}^T \begin{bmatrix} 9 & 1 & 3 \\ 1 & 1 & 0 \\ 3 & 0 & 4 \end{bmatrix} \boldsymbol{w} = \boldsymbol{w}^T \begin{pmatrix} 0 \\ 8 \\ -3 \end{pmatrix} = 72$$

so

$$\mathbf{K} = \begin{bmatrix} 27 & -9 \\ -9 & 72 \end{bmatrix}.$$

(d) The vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ are linearly independent, so the Gram matrix is positive definite. Since there are only two vectors, the only way they could be linearly dependent is if one is a scalar multiple of the other, which is clearly not the case. Another way of establishing linear independence is to set $\boldsymbol{v}$ and $\boldsymbol{w}$ as columns of a matrix, then row reduce to find the rank of the matrix. In this case the rank is 2, which equals the number of columns, which means that the vectors are linearly independent.

