Write your name below. This exam is worth 100 points. On each problem, you must show all your work to receive credit on that problem. You are allowed to use two pages of notes (one piece of paper, front and back). You are not allowed to use a calculator or any computational software.

Name: Section:
(Chi/9:05/001; Grooms/11:15/002; Grooms/1:25/003)

1. (28 points: 4 each) If the statement is **always true** mark "TRUE"; if it is possible for the statement to be false then mark "FALSE". **No justification is necessary**.

_____(a) The polynomials $1 - x + 3x^2, x + x^2, x^2$ span the space of polynomials of degree at most 2.

_____(b) The columns of a matrix $\mathbf{A}$ are linearly independent if and only if the only solution to the homogeneous linear system $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ is the trivial one $\boldsymbol{x} = \mathbf{0}$.

_____(c) Let $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_N$ be vectors in a vector space V, and suppose that the only way to set $\sum_{n=1}^{N} x_n \boldsymbol{v}_n = \mathbf{0}$ is for all the values x_n to be equal. True or false: The dimension of the span of $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_N$ is $N - 1$.

_____(d) The function $F : \mathbb{R}^3 \to \mathbb{R}$ below

$$F(x, y, z) = x - y - z$$

is linear.

_____(e) The following function $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$

$$\langle \mathbf{v}, \mathbf{w} \rangle = v_1 w_1 - v_1 w_2 - v_2 w_1 + 4 v_2 w_2$$

is *not* an inner product.

_____(f) The matrix $\mathbf{K} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ is positive definite.

_____(g) The set of positive definite $n \times n$ matrices is a subspace of the set of $n \times n$ matrices.

2. (32 points, 8 each) Consider the following matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 4 & 0 \\ 2 & 2 & 0 \\ 0 & 2 & 0 \end{pmatrix}$$

(a) What is a basis for range($\mathbf{A}$)? What is its dimension? Explain your answer.

(b) What is a basis for corange($\mathbf{A}$)? What is its dimension? Explain your answer.

(c) What is a basis for kernel($\mathbf{A}$)? What is its dimension? Explain your answer.

(d) What is the dimension of cokernel($\mathbf{A}$)? Explain your answer.

3. (12 points, 6 each) Find the matrix form of the linear transformation

$$L(x, y) = \begin{pmatrix} x - 4y \\ -2x + 3y \end{pmatrix}$$

   (a) with respect to the standard basis of $\mathbb{R}^2$: $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and

   (b) with respect to the basis of $\mathbb{R}^2$: $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

4. (28 points, 7 each points) Let $\boldsymbol{v} = \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$ and $\boldsymbol{w} = \begin{pmatrix} -1 \\ 9 \\ 0 \end{pmatrix}$.

   (a) Find all vectors that are orthogonal to both $\boldsymbol{v}$ and $\boldsymbol{w}$ when orthogonality is defined with respect to the dot product.

   (b) Find all vectors that are orthogonal to both $\boldsymbol{v}$ and $\boldsymbol{w}$ when orthogonality is defined with respect to the inner product

$$\langle \boldsymbol{a}, \boldsymbol{b} \rangle = \boldsymbol{a}^T \begin{bmatrix} 9 & 1 & 3 \\ 1 & 1 & 0 \\ 3 & 0 & 4 \end{bmatrix} \boldsymbol{b}$$

   (c) Find the Gram matrix formed from $\boldsymbol{v}$ and $\boldsymbol{w}$ using the inner product from part (b).

   (d) Is the Gram matrix from part (c) positive definite? Explain why or why not.

