Write your name below. This exam is worth 100 points. On each problem, you must show all your work to receive credit on that problem. You are allowed to use two pages of notes (one piece of paper, front and back). You are not allowed to use a calculator or any computational software.

Name:

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1. (28 points: 4 each) If the statement is **always true** mark "TRUE"; if it is possible for the statement to be false then mark "FALSE". **No justification is necessary**.

_____(a) Suppose you have a coefficient matrix $\mathbf{A}$ with right-hand side $\mathbf{b}$ and an augmented matrix $(\mathbf{A}|\mathbf{b})$ that has been reduced to row echelon form. If there are no rows of the form $\begin{pmatrix} 0 & \cdots & 0 & | & c \neq 0 \end{pmatrix}$ and there are no free variables, then there are infinitely many solutions for the system.

Solution: **False**. Since there are no rows of the form $\begin{pmatrix} 0 & \cdots & 0 & | & c \neq 0 \end{pmatrix}$, the system is compatible. Since there are no free variables, the solution is unique.

_____(b) All permutation matrices $\mathbf{P}$ are idempotent (idempotent means $\mathbf{P}^2 = \mathbf{P}$).

Solution: **False**. Consider $P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$. Then $P^2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ so P is not idempotent.

_____(c) Every nonsingular matrix can be written as the product of elementary matrices.

Solution: **True**. The Gauss-Jordan method uses Gaussian elimination to compute the inverse of a nonsingular matrix. The method first reduces the input matrix A to upper triangular form and then further reduces it to the identity matrix I. Since each of the Gaussian elimination steps used to reduce A to I can be written as an elementary matrix, the product of those elementary matrices gives us the inverse of A.

_____(d) Supposing that all matrices in the expression are the same size and are invertible, $(\mathbf{A}^T\mathbf{B}\mathbf{C}^{-1})^{-1} = \mathbf{C}\mathbf{B}^{-1}\mathbf{A}^T$.

Solution: **False**. The correct expression is $\mathbf{C}\mathbf{B}^{-1}A^{-T}$.

_____(e) Let $\mathbf{A}$ be a square matrix and c be a number. True or False; $\det(c\mathbf{A}) = c\det(\mathbf{A})$.

Solution: **False**. The correct expression is $\det(c\mathbf{A}) = c^n\det(\mathbf{A})$.

_____(f) If $\mathbf{A}$ is a nonsingular matrix then $\mathbf{A}^4 = \mathbf{AAAA}$ is also nonsingular.

Solution: **True**. When A is invertible, the inverse of A^4 is $A^{-4} = A^{-1}A^{-1}A^{-1}A^{-1}$.

_____(g) If $\mathbf{J}$ is an invertible skew-symmetric matrix then $\mathbf{J}^{-1}$ is a symmetric matrix. (Recall that a skew-symmetric matrix is defined by the property that $\mathbf{J}^T = -\mathbf{J}$.)

Solution: **False**. If an invertible matrix is skew-symmetric then its inverse is also skew-symmetric.

2. (32 points, 8 each)

Consider the following system.

$$\begin{aligned} 2x_1 - 6x_2 + 4x_3 &= 2 \\ -x_1 + 3x_2 - 2x_3 &= -1 \end{aligned}$$

(a) Is the system compatible? Why or why not?

(b) Are there any free variables in the system? If so, which variables are free?

(c) What is the rank of the coefficient matrix?

(d) How many solutions are there? If there is a solution, give the solution. If the solution is not unique, give the general solution.

Solution:

(a) We first obtain a row echelon form for the system. (Note that row echelon form is not unique so others are possible.)

$$M = \begin{pmatrix} 2 & -6 & 4 & | & 2 \\ -1 & 3 & -2 & | & -1 \end{pmatrix}$$

Dividing the first row throughout by 2:

$$M \to \begin{pmatrix} 1 & -3 & 2 & | & 1 \\ -1 & 3 & -2 & | & -1 \end{pmatrix}$$

Eliminating below first pivot:

$$M \to \begin{pmatrix} 1 & -3 & 2 & | & 1 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

Since there are no equations of the form $\begin{pmatrix} 0 & \cdots & 0 & | & c \neq 0 \end{pmatrix}$, the system is compatible.

(b) Yes, x_2 and x_3 are free variables since there are no pivots in the corresponding columns.

(c) Since the system is compatible and there are free variables, the system has infinitely many solutions. To get a general solution, we have: $x_1 - 3x_2 + 2x_3 = 1 \quad \Rightarrow \quad x_1 = 1 + 3x_2 - 2x_3$.

So $\mathbf{x} = \begin{pmatrix} 1 + 3x_2 - 2x_3 \\ x_2 \\ x_3 \end{pmatrix}$ is the general solution.

3. (20 points) Let $\boldsymbol{x}$ be a column vector of length m and $\boldsymbol{y}$ be a column vector of length n. You may assume that the first element of each vector is nonzero; all other elements might be zero or nonzero. Prove that the rank of the matrix $\boldsymbol{x}\boldsymbol{y}^T$ is one. (Hint: Try to row reduce the matrix.)

   Solution:

$$xy^T = \begin{bmatrix} x_1y_1 & x_1y_2 & \cdots & x_1y_n \\ x_2y_1 & x_2y_2 & \cdots & x_2y_n \\ \vdots & \vdots & & \vdots \\ x_my_1 & x_my_2 & \cdots & x_my_n \end{bmatrix}.$$

   The first diagonal element is nonzero by assumption (stated in the problem), so we can use it to eliminate all entries in the first column below the first diagonal. Eliminate the second entry in the first column by subtracting x_2/x_1 times the first row from the second row, which yields

$$xy^T = \begin{bmatrix} x_1y_1 & x_1y_2 & \cdots & x_1y_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ x_my_1 & x_my_2 & \cdots & x_my_n \end{bmatrix}.$$

   Inspired by the success of this first step, we eliminate the first entry in row j by subtracting x_j/x_1 times the first row from the j^{th} row. When applied to all the rows this yields

$$xy^T = \begin{bmatrix} x_1y_1 & x_1y_2 & \cdots & x_1y_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

   At this point we have row-reduced the matrix and can see that there is only one pivot, so the rank is 1.

4. Let

$$A = LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & \frac{3}{2} & 1 & 0 \\ \frac{1}{4} & -\frac{1}{8} & \frac{1}{4} & 1 \end{bmatrix} \begin{bmatrix} 1 & 77 & 42 & 10 \\ 0 & e & 4 & 20 \\ 0 & 0 & \frac{\pi}{e} & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(a) (6 points) Find $\det(A)$.

(b) (14 points) Find L^{-1}.

Solution:

(a) The determinant of A is the product of the determinants of L and U. Each of them is a triangular matrix, so their determinants are the products of their diagonal elements. In the case of L this is 1, while in the case of U it is π. The determinant of A is therefore π.

(b) To find L^{-1} we use Gauss-Jordan elimination.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 3 & \frac{3}{2} & 1 & 0 & 0 & 0 & 1 & 0 \\ \frac{1}{4} & -\frac{1}{8} & \frac{1}{4} & 1 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & 1 & 0 & -3 & 0 & 1 & 0 \\ 0 & -\frac{1}{8} & \frac{1}{4} & 1 & -\frac{1}{4} & 0 & 0 & 1 \end{array}\right] \to$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -\frac{3}{2} & 1 & 0 \\ 0 & 0 & \frac{1}{4} & 1 & -\frac{1}{2} & \frac{1}{8} & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -\frac{3}{2} & 1 & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{4} & 1 \end{array}\right]$$

We conclude that

$$L^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & -\frac{3}{2} & 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{4} & 1 \end{bmatrix}.$$

