Write your name below. This exam is worth 100 points. On each problem, you must show all your work to receive credit on that problem. You are allowed to use two pages of notes (one piece of paper, front and back). You are not allowed to use a calculator or any computational software.

Name:

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1. (28 points: 4 each) If the statement is **always true** mark "TRUE"; if it is possible for the statement to be false then mark "FALSE". **No justification is necessary**.

_____(a) Suppose you have a coefficient matrix $\mathbf{A}$ with right-hand side $\mathbf{b}$ and an augmented matrix $(\mathbf{A}|\mathbf{b})$ that has been reduced to row echelon form. If there are no rows of the form $\begin{pmatrix} 0 & \cdots & 0 & | & c \neq 0 \end{pmatrix}$ and there are no free variables, then there are infinitely many solutions for the system.

_____(b) All permutation matrices $\mathbf{P}$ are idempotent (idempotent means $\mathbf{P}^2 = \mathbf{P}$).

_____(c) Every nonsingular matrix can be written as the product of elementary matrices.

_____(d) Supposing that all matrices in the expression are the same size and are invertible, $(\mathbf{A}^T\mathbf{B}\mathbf{C}^{-1})^{-1} = \mathbf{C}\mathbf{B}^{-1}\mathbf{A}^T$.

_____(e) Let $\mathbf{A}$ be a square matrix and c be a number. True or False; $\det(c\mathbf{A}) = c\det(\mathbf{A})$.

_____(f) If $\mathbf{A}$ is a nonsingular matrix then $\mathbf{A}^4 = \mathbf{AAAA}$ is also nonsingular.

_____(g) If $\mathbf{J}$ is an invertible skew-symmetric matrix then $\mathbf{J}^{-1}$ is a symmetric matrix. (Recall that a skew-symmetric matrix is defined by the property that $\mathbf{J}^T = -\mathbf{J}$.)

2. (32 points, 8 each)

   Consider the following system.

$$\begin{aligned} 2x_1 - 6x_2 + 4x_3 &= 2 \\ -x_1 + 3x_2 - 2x_3 &= -1 \end{aligned}$$

   (a) Is the system compatible? Why or why not?

   (b) Are there any free variables in the system? If so, which variables are free?

   (c) What is the rank of the coefficient matrix?

   (d) How many solutions are there? If there is a solution, give the solution. If the solution is not unique, give the general solution.

3. (20 points) Let $\boldsymbol{x}$ be a column vector of length m and $\boldsymbol{y}$ be a column vector of length n. You may assume that the first element of each vector is nonzero; all other elements might be zero or nonzero. Prove that the rank of the matrix $\boldsymbol{x}\boldsymbol{y}^T$ is one. (Hint: Try to row reduce the matrix.)

4. Let

$$A = LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & \frac{3}{2} & 1 & 0 \\ \frac{1}{4} & -\frac{1}{8} & \frac{1}{4} & 1 \end{bmatrix} \begin{bmatrix} 1 & 77 & 42 & 10 \\ 0 & e & 4 & 20 \\ 0 & 0 & \frac{\pi}{e} & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

   (a) (6 points) Find $\det(A)$.

   (b) (14 points) Find L^{-1}.

