Write your name below. You must show your work and not give decimal answers (i.e. don't use a calculator or software to compute a decimal answer). You are not allowed to collaborate on the exam or seek outside help, though using your notes, the book, the recorded lectures, or material you find online is acceptable (you can't ask someone for help online). To receive full credit on a problem you must show **sufficient justification for your conclusion** unless explicitly stated otherwise. Please submit this exam to the course canvas page by **December 13 at 11:59PM (Mountain Time)**.

Name:

1. (20 points: 2 each) If the statement is **always true** mark "TRUE" and provide a *brief* justification; if it is possible for the statement to be false then mark "FALSE" and provide a counterexample or the correct statement.

   (a) If $\mathbf{A}$ is a 4×2 matrix and $\mathbf{B}$ is a 3×4 matrix, then the product $\mathbf{BA}$ is defined.

   Solution: True. The inner matrix dimensions match, so the matrix product is defined.

   (b) $\mathbf{A}$ is nonsingular if and only if $\mathbf{A}$ has an eigenvalue 0. (You may assume that $\mathbf{A}$ is square.)

   Solution: False. Ex A = 0 has an eigenvalue 0 and is singular. Just checking vocabulary.

   (c) If $\mathbf{A} = \mathbf{TDT}^{-1}$ for some diagonal matrix $\mathbf{D}$ and invertible matrix $\mathbf{T}$, then the columns of $\mathbf{T}$ are eigenvectors of $\mathbf{A}$.

   Solution: True. Re-write as $\mathbf{AT} = \mathbf{TD}$, then write out column by column: $\mathbf{A}\vec{t}_1 = d_1\vec{t}_1, \ldots, \mathbf{A}\vec{t}_n = d_n\vec{t}_n$.

(d) Let $\mathbf{A} = \mathbf{P}\boldsymbol{\Sigma}\boldsymbol{Q}^T$ be the SVD of $\mathbf{A}$. True or false: The columns of $\mathbf{P}$ are eigenvectors of $\mathbf{A}\mathbf{A}^T$.

Solution: True. $\mathbf{A}\mathbf{A}^T = \mathbf{P}\boldsymbol{\Sigma}^2\mathbf{P}^T$ is the spectral decomposition of $\mathbf{A}\mathbf{A}^T$, so the columns of $\mathbf{P}$ are eigenvectors.

(e) If $\mathbf{A}$ has full column rank and $\mathbf{A}^\dagger$ is the pseudoinverse of $\mathbf{A}$, then $\mathbf{A}\mathbf{A}^\dagger$ is an orthogonal projection matrix that projects onto the range of $\mathbf{A}$.

Solution: True. The usual formula is $\mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T$, and when $\mathbf{A}$ has full column rank we recognize $(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T = \mathbf{A}^\dagger$.

(f) If $\mathbf{A}$ is an $m \times n$ matrix and the linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$ has two free variables, then $\text{rank}(A) = n$.

Solution: False Since n would be the sum of the rank and the dimension of the kernal and the kernel has dimension 2, we know that the rank must be $n - 2$.

(g) If λ is an eigenvalue of matrix $\mathbf{A}$, then λ^2 is an eigenvalue of $\mathbf{A}^2$.

Solution: True. Suppose $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$ for nonzero $\mathbf{v}$. Then, $\mathbf{A^2}\mathbf{v} = \lambda\mathbf{A}\mathbf{v} = \lambda^2\mathbf{v}$.

(h) Suppose $\mathbf{A}$ is a symmetric matrix. Then, $\mathbf{Ax} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is orthogonal to the corange of $\mathbf{A}$.

Solution: False. Consider the symmetric matrix $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and the vector $\mathbf{b} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Note that $\mathbf{b}$ is orthogonal to the corange of $\mathbf{A}$ but that $\mathbf{Ax} = \mathbf{b}$ has no solution.

(i) If $\mathbf{A}$ is invertible, then $\mathbf{A}$ is diagonalizable.

Solution: False. The matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is invertible but not diagonalizable.

(j) The singular values of a nonsingular matrix $\mathbf{A}$ are the same as the singular values of $\mathbf{A}^{-1}$.

Solution: False. The matrix $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ has only the singular value 2, but it's inverse has only the singular value $\frac{1}{2}$.

2. Consider the following linear transformation: $L(x, y) = \begin{pmatrix} -x + y \\ 2x - y \end{pmatrix}$.

   (a) (4 points) Find the matrix form of $L(x, y)$ with respect to the standard basis.

   Solution is obtained by plugging the standard basis vectors in, and then setting the resulting output as columns of $\mathbf{A}$:

   $$\begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$$

   (b) (6 points) Find the matrix form of $L(x, y)$ with respect to the following basis: $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. (The same basis is used for both the domain space and the co-domain space.)

   Solution is obtained by constructing the matrix $\mathbf{S} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and then finding $\mathbf{S}^{-1}\mathbf{AS}$. The result is

   $$\begin{pmatrix} -3 & -1 \\ 2 & 1 \end{pmatrix} \text{ and note that } \mathbf{S}^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

3. (20 points) Find the Jordan decomposition of $\mathbf{A} = \begin{bmatrix} -3 & 3 & -3 \\ -1 & -1 & -1 \\ -2 & 1 & -2 \end{bmatrix}$.

Solution: First find the eigenvalues. The characteristic equation is $-\lambda(\lambda+3)^2 = 0$, so the eigenvalues are $\lambda = 0$ and $\lambda = -3$ (double).

Next find the eigenvectors by row reducing $\mathbf{A} - \lambda\mathbf{I}$. The REF and eigenvector for $\lambda = 0$ are

$$\text{REF } = \begin{bmatrix} -3 & 3 & -3 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ \vec{v}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

The REF and eigenvector for $\lambda = -3$ are

$$\text{REF } = \begin{bmatrix} -1 & 2 & -1 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{bmatrix}, \ \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

(There is more than one REF depending on how you pivot, but the eigenvector is always the same.)

Since there is one defective eigenvalue $\lambda = -3$ we look for a Jordan chain that branches from $\vec{v}_2$ by computing the particular solution of $(\mathbf{A} + 3\mathbf{I})\vec{w}_2 = \vec{v}_2$. The REF of the augmented matrix is

$$\text{REF } = \left[\begin{array}{ccc|c} -1 & 2 & -1 & 1 \\ 0 & 3 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The particular solution is obtained by setting the free variable to 0, which yields $\vec{w}_2 = \frac{1}{3}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$. The Jordan decomposition is $\mathbf{A} = \mathbf{S}\mathbf{J}\mathbf{S}^{-1}$. $\mathbf{S}^{-1}$ can be found using Gauss-Jordan, but this is worth only 2 points. There are multiple answers depending on the arrangement of the Jordan blocks. These are:

$$\mathbf{S} = \begin{bmatrix} 1 & -\frac{1}{3} & -1 \\ 1 & \frac{1}{3} & 0 \\ 1 & 0 & 1 \end{bmatrix}, \ \mathbf{J} = \begin{bmatrix} -3 & 1 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ \mathbf{S}^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -1 & 2 & -1 \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \text{ or}$$

$$\mathbf{S} = \begin{bmatrix} -1 & 1 & -\frac{1}{3} \\ 0 & 1 & \frac{1}{3} \\ 1 & 1 & 0 \end{bmatrix}, \ \mathbf{J} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{bmatrix}, \ \mathbf{S}^{-1} = \begin{bmatrix} -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -1 & 2 & -1 \end{bmatrix}$$

4. (20 points) Find the singular value decomposition of

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}.$$

Solution:

First, find SVD of $\mathbf{B} = \mathbf{A}^T$.

The eigenvalues of $\mathbf{B}^T\mathbf{B}$ are $\lambda_1 = 25$ and $\lambda_2 = 9$, so the singular values of $\mathbf{B}$ (and therefore $\mathbf{A}$) are $\sigma_1 = 5$ and $\sigma_2 = 3$. So, let $\mathbf{\Sigma} = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$.

Corresponding eigenvectors of $\mathbf{B}^T\mathbf{B}$ of unit length are $\mathbf{q}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{q}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. So, let $\mathbf{Q} = [\mathbf{q}_1\mathbf{q}_2] = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

Let $\mathbf{p}_1 = \frac{1}{\sigma_1}\mathbf{B}\mathbf{q}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{p}_2 = \frac{1}{\sigma_2}\mathbf{B}\mathbf{q}_2 = \frac{1}{3\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$. So, we define $\mathbf{P} = [\mathbf{p}_1\mathbf{p}_2] = \frac{1}{3\sqrt{2}}\begin{bmatrix} 3 & 1 \\ 3 & -1 \\ 0 & 4 \end{bmatrix}$.

So, the SVD of $\mathbf{B}$ is $\mathbf{B} = \mathbf{P\Sigma Q}^T$, which means the SVD of $\mathbf{A}$ is

$$\mathbf{A} = \mathbf{Q\Sigma P}^T = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}\left(\frac{1}{3\sqrt{2}}\begin{bmatrix} 3 & 3 & 0 \\ 1 & -1 & 4 \end{bmatrix}\right).$$

5. (10 points) Find the LU factorization of the following matrix. Do not use row permutations.

$$\begin{bmatrix} 2 & 1 & 0 \\ -4 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

Solution is

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{bmatrix}, \ \mathbf{U} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

6. (10 points) Find bases for both the range and corange of the following matrix. (Be sure your answer is supported by your work.)

$$\mathbf{A} = \begin{bmatrix} 1 & -3 & -3 \\ -2 & 4 & 2 \\ -1 & 5 & 7 \end{bmatrix}.$$

Solution

If we row reduce $\mathbf{A}$, we see

$$\mathbf{A} \to \begin{bmatrix} 1 & -3 & -3 \\ 0 & -2 & -4 \\ 0 & 2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & -3 \\ 0 & -2 & -4 \\ 0 & 0 & 0 \end{bmatrix}.$$

Noting the positions of the pivots, the corresponding columns of $\mathbf{A}$ form a basis for the range of $\mathbf{A}$: $\left\{ \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 5 \end{bmatrix} \right\}$. Also, the rows of the reduced form of $\mathbf{A}$ give us a basis for the corange of $\mathbf{A}$: $\left\{ \begin{bmatrix} 1 \\ -3 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ -4 \end{bmatrix} \right\}$.

7. (10 points) Find the least-squares solution of the system

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 2 \\ 5 \\ 6 \\ 6 \end{bmatrix}.$$

Solution:

Write

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix}$$

and

$$\mathbf{b} = \begin{bmatrix} 2 \\ 5 \\ 6 \\ 6 \end{bmatrix}.$$

Then we find that $\mathbf{A}^T\mathbf{A} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ and $\mathbf{A}^T\mathbf{b} = \begin{bmatrix} 1 \\ 14 \\ -5 \end{bmatrix}$. The least square solution of $\mathbf{Ax} = \mathbf{b}$ is the solution of the normal equation $\mathbf{A^T A x} = \mathbf{A^T b}$ which is

$$\mathbf{x} = \frac{1}{3}\begin{bmatrix} 1 \\ 14 \\ -5 \end{bmatrix}.$$