Write your name and your professor's name or below. You must show your work and not give decimal answers (i.e. dont use a calculator or software to compute a decimal answer). You are not allowed to collaborate on the exam or seek outside help, though using your notes, the book, the recorded lectures, or material you find online is acceptable (you cant ask someone for help online). To receive full credit on a problem you must show **sufficient justification for your conclusion** unless explicitly stated otherwise. Please submit this exam to the course canvas page by **October 21 at 11:59PM (Mountain Time)**.
Name:
Instructor/Section:

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1. (30 points: 5 each) If the statement is **always true** mark "TRUE" and provide a *brief* justification; if it is possible for the statement to be false then mark "FALSE" and provide a counterexample.

   (a) Suppose that W is a nonempty subset of another set V. In other words, $W \subset V$ and W is not the empty set, $\{\}$. Then W is a *subspace* of V if the following two features are true:

   (i) for every $\mathbf{v}, \mathbf{w} \in W$, the sum $\mathbf{v} + \mathbf{w} \in W$, and

   (ii) for every $\mathbf{v} \in W$ and every $c \in \mathbb{R}$, the scalar product $c\mathbf{v} \in W$.

   False. We also need to establish that V is actually a vector space. This requires checking nine properties of V.

   (b) The vectors $\mathbf{u} = (1, 2, -1)^T$, $\mathbf{v} = (3, 1, 5)^T$, and $\mathbf{w} = (8, 1, 2)^T$ are linearly dependent.

   False. Let $A = (\mathbf{u}\ \mathbf{v}\ \mathbf{w})$. Theorem 2.21 on page 95 of the text tells us that the vectors are linearly dependent if and only if there is a non-zero solution $\mathbf{c} \neq \mathbf{0}$ to the homogeneous linear system $A\mathbf{c} = \mathbf{0}$. Gaussian elimination turns A into the following matrix:

   $$\begin{pmatrix} 1 & 3 & 8 \\ 0 & -5 & 15 \\ 0 & 0 & 14 \end{pmatrix}.$$

   This has rank 3, so the only solution to $A\mathbf{c} = \mathbf{0}$ is the zero vector.

   (c) The columns of the following matrix span $\mathbb{R}^3$:

   $$\mathbf{A} = \begin{bmatrix} 3 & 4 & -1 \\ -2 & 7 & 3 \\ 3 & 2 & -1/2 \end{bmatrix}.$$

   True. Theorem 2.28 tells us that the columns for a basis for $\mathbb{R}^3$ if and only if rank $A = 3$. Since rank $A = 3$, the columns are a basis and span $\mathbf{R}^3$.

(d) If $\mathbf{A}$ is an $m \times n$ matrix and the linear system $\mathbf{Ax} = \mathbf{b}$ has no free variables, then $\text{rank}(A) = n$.

True. See Proposition 2.41 on page 108 of the text.

(e) Let $\mathbf{A}$ be the following 3×3 matrix:

$$\begin{bmatrix} 1 & 0 & -1 \\ 4 & 3 & 2 \\ 2 & 1 & 3 \end{bmatrix}.$$

Then $||\mathbf{A}||_\infty = 7$.

False. The infinity norm of A is the maximum row sum of the absolute values of A,

$$||\mathbf{A}||_\infty = \max_{1 \leq i \leq 3} \sum_{j=1}^{3} |a_{ij}|.$$

This occurs in row 2. $4 + 3 + 2 = 9$.

(f) It is always true that $||\mathbf{A} + \mathbf{B} + \mathbf{C}|| \leq ||\mathbf{A}|| + ||\mathbf{B}|| + ||\mathbf{C}||$.

True. Apply the triangle inequality twice. For example,

$$||\mathbf{A} + \mathbf{B} + \mathbf{C}|| \leq ||\mathbf{A} + \mathbf{B}|| + ||\mathbf{C}| \leq ||\mathbf{A}|| + ||\mathbf{B}|| + ||\mathbf{C}||.$$

2. Let $\mathbf{A}=\begin{bmatrix} 4 & 0 & 5 \\ 2 & 3 & -1 \\ -2 & 1 & 0 \end{bmatrix}$.

(a) (10 points) Find a basis for the *kernel* of the columns of **A**.

Row reduce to find

$$\begin{bmatrix} 4 & 0 & 5 \\ 2 & 3 & -1 \\ -2 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 4 & 0 & 5 \\ 0 & 3 & -7/2 \\ 0 & 1 & 5/2 \end{bmatrix} \to \begin{bmatrix} 4 & 0 & 5 \\ 0 & 3 & -7/2 \\ 0 & 0 & 11/3 \end{bmatrix}.$$

The only solution to $A\mathbf{z} = \mathbf{0}$ is the zero vector. Therefore,

$$\ker A = \{\mathbf{0}\}.$$

Notice that technically there is no basis for the kernel. Saying that $\mathbf{0}$ is a basis is technically incorrect, because (i) it would imply that the dimension is 1 (number of basis vectors), and (ii) $\mathbf{0}$ can never be an element of a basis. No points were taken off for these kinds of errors.

(b) (10 points) Find a basis for the *range* of **A**.

We know from part (a), that any three linearly independent vectors in $\mathbb{R}^3$ form a basis for the range of A. You can choose the columns of A as a basis, or the standard basis, or any other basis as long as you justify your answer.

3. Let $\mathbf{A} = \begin{bmatrix} 0 & 1 & -3 \\ 0 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix}$.

   (a) (10 points) What is the Frobenius norm of **A**?

   The Frobenius norm of an $m \times n$ matrix A is

   $$||A||_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2}.$$

   In this case,

   $$||A||_F = \sqrt{1^2 + (-3)^2 + 2^2 + 3^2 + 1^2 + 2^2} = \sqrt{28} = 2\sqrt{7}.$$

   (b) (10 points) What is the 1-norm of **A**?

   The maximum absolute column sum is the $1-$norm of A. In this case, it comes from the third column. ‘

   $$||A||_1 = |-3| + |3| + |2| = 8.$$

4. Consider the matrix $\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

   (a) (10 points) Is $\mathbf{A}$ *symmetric positive definite* ?

   A is symmetric. To check if it is SPD, we check that we can row reduce without pivoting, and all the pivots are positive. Row reducing without pivoting yields

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \to \begin{bmatrix} 2 & -1 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & -1 & 2 \end{bmatrix} \to \begin{bmatrix} 2 & -1 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & 0 & \frac{4}{3} \end{bmatrix}.$$

   Therefore, A is symmetric positive definite.

   Alternative Solution: First, note that $\mathbf{A}$ is symmetric. Then, notice that for a vector $\mathbf{x}$, we have

$$\mathbf{x^T A x} = 2x_1^2 - 2x_1x_2 + 2x_2^2 - 2x_2x_3 + x_3^2 = (x_1 - x_2)^2 + (x_2 - x_3)^2 + x_1^2 + x_3^2 \geq 0$$

   where equality can only hold if $\mathbf{x} = \mathbf{0}$. So, A is symmetric positive definite.

   (b) (10 points) Is $\mathbf{A^T}$ *symmetric positive definite*?

   Since $A^T = A$, it is also symmetric positive definite.

5. (10 points) Find the angle between the following vectors using the standard dot product:

$$\begin{bmatrix} 2 & -1 & 0 \end{bmatrix}^T, \quad \text{and} \quad \begin{bmatrix} -1 & 2 & -1 \end{bmatrix}^T.$$

The angle between these vectors is given by

$$\cos(\theta) = \frac{< \mathbf{v}, \mathbf{w} >}{||\mathbf{v}||||\mathbf{w}||}.$$

In this case we have

$$\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} = -2 - 2 + 0 = -4,$$

$$|| \begin{bmatrix} 2 & -1 & 0 \end{bmatrix}^T || = \sqrt{2^2 + (-1)^2} = \sqrt{5},$$

and

$$|| \begin{bmatrix} -1 & 2 & -1 \end{bmatrix}^T || = \sqrt{(-1)^2 + 2^2 + (-1)^2} = \sqrt{6}.$$

Therefore,

$$\cos(\theta) = \frac{-4}{\sqrt{5}\sqrt{6}}$$

or, alternatively,

$$\theta = \cos^{-1}\left(\frac{-4}{\sqrt{5}\sqrt{6}}\right).$$