Write your name and your professor's name below. You must show your work and not give decimal answers (i.e. don't use a calculator or software to compute a decimal answer). You are not allowed to collaborate on the exam or seek outside help, though using your notes, the book, the recorded lectures, or material you find online is acceptable (you can't ask someone for help online). To receive full credit on a problem you must show **sufficient justification for your conclusion** unless explicitly stated otherwise. Please submit this exam to the course canvas page by **September 23 at 11:59PM (Mountain Time)**.

Name:

Instructor:

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1. (30 points: 5 each) If the statement is **always true** mark "TRUE" and provide a *brief* justification; if it is possible for the statement to be false then mark "FALSE" and provide a counterexample.

   (a) A linear system of equations consisting of the same number of variables and equations always has a unique solution.

   False. As a counterexample, consider the system

$$\begin{cases} x_1 + x_2 + x_3 = 1 \\ 2x_1 + 2x_2 + 2x_3 = 2 \\ 3x_1 + 3x_2 + 3x_3 = 3 \end{cases}$$

   which will feature two free variables and have infinitely many solutions.

   (b) The determinant of a nonsingular matrix is 0.

   False. A *singular* matrix will have determinant 0. As a counterexample to the given statement, note that the identity matrix is nonsingular and has a determinant of 1.

   (c) If $\mathbf{A}$ is a 4×3 matrix and $\mathbf{B}$ is a 2×4 matrix, then the product $\mathbf{BA}$ is defined.

   True. Since $\mathbf{B}$ has the same number columns as $\mathbf{A}$ has for rows, then the product $\mathbf{BA}$ is defined.

(d) If a 2×2 upper-triangular matrix is invertible, then its inverse is also upper-triangular.

True. An arbitrary 2×2 upper-triangular matrix is of the form: $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$. If it is invertible, then its inverse is $\frac{1}{ad}\begin{bmatrix} d & -b \\ 0 & a \end{bmatrix}$.

(e) If $\mathbf{Ax} = \mathbf{b}$ is incompatible, then so is $\mathbf{Ax{=}c}$.

False. Suppose $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and $\mathbf{c} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Then $\mathbf{Ax} = \mathbf{b}$ is incompatible but $\mathbf{Ax{=}c}$ is compatible.

(f) If A is invertible, then the determinant of $\mathbf{A}$ is the same as the determinant of $\mathbf{A}^{-1}$.

False. As a counterexample, if $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$, then $\mathbf{A}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{bmatrix}$. We see that the determinant of $\mathbf{A}$ is 4, but the determinant of $\mathbf{A}^{-1}$ is $\frac{1}{4}$.

2. Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 3 & 0 & 1 \end{bmatrix}$.

   (a) (10 points) Use Gauss-Jordan Elimination to find the inverse of $\mathbf{A}$.

$$[\mathbf{A}|\mathbf{I}] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 3 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3' = R_3 - 3R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2' = R_2 - 4R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 12 & 1 & -4 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right] = [\mathbf{I}|\mathbf{A^{-1}}]$$

   So,

$$\mathbf{A^{-1}} = \begin{bmatrix} 1 & 0 & 0 \\ 12 & 1 & -4 \\ -3 & 0 & 1 \end{bmatrix}.$$

   (b) (10 points) Use your answer from (a) to find the solution of

$$\begin{cases} x_1 & & = -3 \\ & x_2 + 4x_3 & = 1 \\ 3x_1 & + \; x_3 & = 0 \end{cases}$$

$$\mathbf{x} = \mathbf{A^{-1}b} = \begin{bmatrix} 1 & 0 & 0 \\ 12 & 1 & -4 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ -35 \\ 9 \end{bmatrix}.$$

3. Let $\mathbf{A} = \begin{bmatrix} 0 & 1 & -3 \\ 0 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix}$.

   (a) (10 points) Determine the permuted LU-factorization of **A**.

   First we row-reduce the matrix **A**:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & -3 \\ 0 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 3 \\ 0 & 1 & -3 \end{bmatrix} \xrightarrow{R_3' \to R_3 - \frac{1}{2}R_2} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & -\frac{9}{2} \end{bmatrix} = \mathbf{U}.$$

   The first row operation was the only row interchange, so we have that

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

   The second row operation was the only instance of adding a multiple of one row to another, so we have

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{bmatrix}.$$

   We can multiply to verify that $\mathbf{PA} = \mathbf{LU}$.

   (b) (10 points) What is the determinant of **A**?

   Since we have the permuted LU-factorization from (a) and only one row interchange was required, we see that the determinant of **A** is

$$\begin{aligned} \det(\mathbf{A}) &= (-1)^1 \det(\mathbf{U}) \\ &= -(1)(2)\left(-\frac{9}{2}\right) \\ &= 9. \end{aligned}$$

   Alternatively, we can use Laplace Cofactor Expansion to determine the determinant of **A**.

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(c) (10 points) Use the permuted LU-factorization from (a) to solve the system $\boldsymbol{A}\boldsymbol{x} = (1,5,2)^T$. You must use the permuted LU factorization to receive points. Other methods will receive no points.

Let

$$\mathbf{b} = \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}.$$

We will use the permuted-LU factorization we found in (a) to solve $\mathbf{Ax} = \mathbf{b}$. To do this, recall that we must first solve $\mathbf{Ly} = \mathbf{Pb}$ for $\mathbf{y}$, and then we will be able to solve $\mathbf{Ux} = \mathbf{y}$ for $\mathbf{x}$.

First, we see that

$$\mathbf{Pb} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}.$$

So, in forward-solving $\mathbf{Ly} = \mathbf{Pb}$, that is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix},$$

we see that $y_1 = 2$, $y_2 = 5$, and $\frac{1}{2}(5) + y_3 = 1$. This last equation yields $y_3 = -\frac{3}{2}$. So, we have

$$\mathbf{y} = \begin{bmatrix} 2 \\ 5 \\ -\frac{3}{2} \end{bmatrix}.$$

Now, we back-solve to solve $\mathbf{Ux} = \mathbf{y}$ for $\mathbf{x}$. We have

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & -\frac{9}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ -\frac{3}{2} \end{bmatrix},$$

so we see that $-\frac{9}{2}x_3 = -\frac{3}{2}$. Thus, $x_3 = \frac{1}{3}$. From the second row, we see that $2x_2 + 3\left(\frac{1}{3}\right) = 5$, which tells us that $x_2 = 2$. Lastly, from the first row we see that $x_1 + 2\left(\frac{1}{3}\right) = 2$, which tells us that $x_3 = \frac{4}{3}$. So, the solution of the equation $\mathbf{Ax} = \mathbf{b}$ is

$$\mathbf{x} = \begin{bmatrix} \frac{4}{3} \\ 2 \\ \frac{1}{3} \end{bmatrix}.$$

4. (10 points) Suppose that $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$ are similar. That is, there exists an invertible matrix $\mathbf{S}$ such that $\mathbf{B} = \mathbf{S}^{-1}\mathbf{AS}$. Prove that $\mathbf{A}$ and $\mathbf{B}$ have the same determinant.

We know that the determinant of the product of matrices is the product of the determinants. We also know that the determinant of an inverse matrix is the reciprocal of the determinant of the original matrix. So, we have the following:

$$\begin{aligned}
\det(\mathbf{B}) &= \det(\mathbf{S^{-1}AS}) \\
&= \det(\mathbf{S^{-1}})\det(\mathbf{A})\det(\mathbf{S}) \\
&= \frac{1}{\det(\mathbf{S})}\det(\mathbf{A})\det(\mathbf{S}) \\
&= \det(\mathbf{A}).
\end{aligned}$$

5. (10 points) Find all solutions of the following system:

$$\begin{cases} x_1 + 4x_2 - 2x_3 = -3 \\ 2x_1 + \ \ x_2 + 3x_3 = 1 \end{cases}$$

We will write this system as an augmented matrix and then row reduce using Gauss-Jordan Elimination:

$$\left[\begin{array}{ccc|c} 1 & 4 & -2 & -3 \\ 2 & 1 & 3 & 1 \end{array}\right] \xrightarrow{R_2'=R_2-2R_1} \left[\begin{array}{ccc|c} 1 & 4 & -2 & -3 \\ 0 & -7 & 7 & 7 \end{array}\right] \xrightarrow{R_2'=-\frac{1}{7}R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 4 & -2 & -3 \\ 0 & 1 & -1 & -1 \end{array}\right] \xrightarrow{R_1'=R_1-4R_2} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & -1 \end{array}\right]$$

From this, we see that x_1 and x_2 are lead variables, and x_3 is a free variable. Specifically, we see that $x_1 = 1 - 2x_3$ and $x_2 = -1 + x_3$. So, our solutions can be written as

$$\mathbf{x} = \begin{bmatrix} 1 - 2x_3 \\ -1 + x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$