- This exam is worth 150 points and has 5 problems.
- Show all work and simplify your answers! Answers with no justification will receive no points unless otherwise noted.
- Begin each problem on a new page.
- **DO NOT LEAVE THE EXAM UNTIL YOUR HAVE SATISFACTORILY SCANNED AND UPLOADED YOUR EXAM TO GRADESCOPE.**
- You are taking this exam in a proctored and honor code enforced environment. No calculators, cell phones, or other electronic devices or the internet are permitted during the exam. You are allowed one 8.5”× 11” crib sheet with writing on both sides.
- Remote students are allowed use of a computer during the exam only for a live video of their hands and face and to view the exam in the Zoom meeting. Remote students cannot interact with anyone except the proctor during the exam.

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0. At the top of the first page that you will be scanning and uploading to Gradescope, write the following statement and sign your name to it: ”I will abide by the CU Boulder Honor Code on this exam.” **FAILURE TO INCLUDE THIS STATEMENT AND YOUR SIGNATURE MAY RESULT IN A PENALTY.**

1. [2360/072624 (24 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given.

   (a) The set of all vectors of the form $\vec{\mathbf{x}} = \begin{bmatrix} a & b & c & abc \end{bmatrix}^{\mathrm{T}}$, $a, b, c \in \mathbb{R}$, is a subspace of $\mathbb{R}^4$.

   (b) $y'' + 2y' + 3y = \ln t,\ t > 0$, can be solved using the method of undetermined coefficients.

   (c) An undamped harmonic oscillator consisting of a mass of 2 kg and having a circular (angular) frequency of 5 sec^{-1} is subject to an external force of 5 N precisely and only at $t = 5$. The differential equation governing the motion is $2\ddot{x} + 50x = 5\delta(t-5)$.

   (d) The following system of equations has no equilibrium solutions.

   $$\begin{aligned} x' &= x^2 + y^4 + 1 \\ y' &= x^4 + y^2 \end{aligned}$$

   (e) The set $\{t^2 + t, 2t, 3t - 1\}$ forms a basis for $\mathbb{P}_2$.

   (f) The equilibrium point of the following system is an unstable spiral: $\vec{\mathbf{x}}' = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} \vec{\mathbf{x}}$.

   (g) If $\mathbf{Q}$ is a square matrix such that $\mathbf{Q}^{-1} = \mathbf{Q}^{\mathrm{T}}$, then $\left(\mathbf{Q}^{\mathrm{T}}\mathbf{Q}\right)^{-1} = \mathbf{Q}^{\mathrm{T}}\mathbf{Q}$.

   (h) The integrating factor for $t^2 y' + 4ty = t^{-2}$ is $\mu(t) = 2t^2$.

2. [2360/072624 (44 pts)] Let $\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

   (a) (6 pts) Is $\mathbf{A}$ invertible? Justify your answer.

   (b) (6 pts) Is $\mathbf{A}$ in RREF? If not, what is its RREF?

   (c) (6 pts) How many solutions does $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{0}}$ have? Justify your answer.

   (d) (6 pts) Is $\vec{\mathbf{b}} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ in col $\mathbf{A}$? Justify your answer.

   (e) (20 pts) Solve the initial value problem $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}},\ \vec{\mathbf{x}}(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Write your answer as a single vector.

MORE PROBLEMS and LAPLACE TRANSFORM TABLE BELOW/ON REVERSE

3. [2360/072624 (40 pts)] A mass/spring system (harmonic oscillator) is governed by the differential equation $\ddot{x} + 8\dot{x} + 25x = f(t)$.

(a) (5 pts) What is $f(t)$ if the oscillator is unforced?

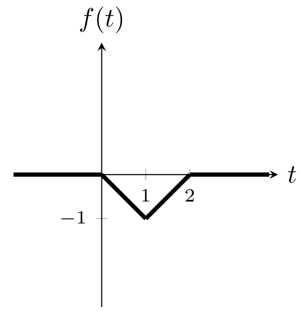
(b) (5 pts) Suppose the system is unforced and the initial displacement is positive. Will the mass pass through its equilibrium position more than once, regardless of the initial velocity? Justify your answer.

(c) (5 pts) Does an $f(t)$ exist such that the total energy of the system remains constant for all time? Justify your answer.

(d) (25 pts) If $f(t) = \begin{cases} 0 & 0 \le t < 2 \\ 25 & t \ge 2 \end{cases}$ and the mass is resting at its equilibrium position when $t = 0$, use Laplace Transforms to find the equation of motion of the oscillator.

4. [2360/072624 (20 pts)] The following problems are not related.

(a) (10 pts) The graph of $f(t)$ is shown in the figure. Write $f(t)$ as a single function, not as a piecewise defined function.



(b) (10 pts) Find the Laplace transform of $g(t) = t\,\text{step}(t-1) + \sin(t-\pi)\,\text{step}(t-\pi) - t\delta(t-4)$

5. [2360/072624 (22 pts)] Consider the differential equation $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{1}{2}(y-1)^3$.

(a) (2 pts) Is the differential equation autonomous? Why or why not?

(b) (3 pts) Find all equilibrium solutions and determine their stability.

(c) (3 pts) Plot the phase line, correctly depicting all equilibrium solutions.

(d) (2 pts) For any initial value y_0, what is $\lim\limits_{x\to\infty} y(x)$? Justify your answer.

(e) (12 pts) Find the explicit form of the solution of the differential equation that passes through the origin.

Short table of Laplace Transforms: $\mathscr{L}\{f(t)\} = F(s) \equiv \displaystyle\int_0^\infty e^{-st} f(t)\,\mathrm{d}t$

In this table, a, b, c are real numbers with $c \ge 0$, and $n = 0, 1, 2, 3, \ldots$

$$\mathscr{L}\left\{t^n e^{at}\right\} = \frac{n!}{(s-a)^{n+1}} \qquad \mathscr{L}\left\{e^{at}\cos bt\right\} = \frac{s-a}{(s-a)^2+b^2} \qquad \mathscr{L}\left\{e^{at}\sin bt\right\} = \frac{b}{(s-a)^2+b^2}$$

$$\mathscr{L}\{\cosh bt\} = \frac{s}{s^2-b^2} \qquad \mathscr{L}\{\sinh bt\} = \frac{b}{s^2-b^2}$$

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{\mathrm{d}^n F(s)}{\mathrm{d}s^n} \qquad \mathscr{L}\left\{e^{at} f(t)\right\} = F(s-a) \qquad \mathscr{L}\{\delta(t-c)\} = e^{-cs}$$

$$\mathscr{L}\{tf'(t)\} = -F(s) - s\frac{\mathrm{d}F(s)}{\mathrm{d}s} \qquad \mathscr{L}\{f(t-c)\,\text{step}(t-c)\} = e^{-cs}F(s) \qquad \mathscr{L}\{f(t)\,\text{step}(t-c)\} = e^{-cs}\mathscr{L}\{f(t+c)\}$$

$$\mathscr{L}\left\{f^{(n)}(t)\right\} = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \cdots - f^{(n-1)}(0)$$
