1. [2360/072823 (25 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given.

   (a) The matrix $\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix}$ is in RREF.

   (b) The subset $\mathbb{W} = \left\{ (m,n) \in \mathbb{R}^2 \,\middle|\, m,n \text{ integers} \right\}$ is a subspace of $\mathbb{R}^2$.

   (c) For any matrix $\mathbf{B}$, $\left|\mathbf{B}\mathbf{B}^{\mathrm{T}}\right|$ is defined.

   (d) The linear system $\mathbf{C}\vec{\mathbf{y}} = \vec{\mathbf{d}}$, where $\mathbf{C}$ is an $n \times n$ matrix, has the unique solution $\vec{\mathbf{x}} = \mathbf{C}^{-1}\vec{\mathbf{d}}$ if and only if the RREF of $\mathbf{C} = \mathbf{I}$.

   (e) $e^{t^2/2}$ is the integrating factor for the differential equation $\dfrac{y'}{t^2} + ty = \sin t$.

SOLUTION:

   (a) **TRUE** All requirements of RREF are met.

   (b) **FALSE** For any non-integer real number c, $c(m,n) = (cm, cn)$ does not have integer coordinates. The subset is not closed under scalar multiplication and therefore not a subspace.

   (c) **TRUE** For any $m \times n$ matrix $\mathbf{B}$ (m not necessarily equal to n), $\mathbf{B}\mathbf{B}^{\mathrm{T}}$ has order $m \times m$ for which the determinant is defined.

   (d) **TRUE** $\mathbf{C}$ is invertible if and only if it is row equivalent to $\mathbf{I}$, that is, its RREF is $\mathbf{I}$.

   (e) **FALSE** To find the integrating factor, the equation must be in the form $y' + t^3 y = t^2 \sin t$, from which the integrating factor is determined to be $e^{t^4/4}$.

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2. [2360/072823 (30 pts)] Find the solution to the initial value problem $\vec{\mathbf{x}}' = \begin{bmatrix} 1 & 1 \\ -4 & 1 \end{bmatrix} \vec{\mathbf{x}}$, $\vec{\mathbf{x}}(0) = \begin{bmatrix} -1 & 2 \end{bmatrix}^{\mathrm{T}}$. Write your final answer as a single vector.

SOLUTION:

$$\begin{vmatrix} 1-\lambda & 1 \\ -4 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 4$$

$$= \lambda^2 - 2\lambda + 1 + 4 = \lambda^2 - 2\lambda + 5 = 0$$

$$\lambda = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i$$

For $\lambda = 1 + 2i$,

$$\left[\begin{array}{cc|c} 1-(1+2i) & 1 & 0 \\ -4 & 1-(1+2i) & 0 \end{array}\right] = \left[\begin{array}{cc|c} -2i & 1 & 0 \\ -4 & -2i & 0 \end{array}\right] \xrightarrow{R_1^* = \frac{i}{2}R_1} \left[\begin{array}{cc|c} 1 & \frac{i}{2} & 0 \\ -4 & -2i & 0 \end{array}\right] \xrightarrow{R_2^* = 4R_1 + R_2} \left[\begin{array}{cc|c} 1 & \frac{i}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

This gives us $\vec{\mathbf{v}} = \begin{bmatrix} -i \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} + i \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ as an eigenvector. Then $\alpha = 1$, $\beta = 2$, $\vec{\mathbf{p}} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ and $\vec{\mathbf{q}} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. This gives the general solution

$$\vec{\mathbf{x}}(t) = c_1 e^t \left(\cos 2t \begin{bmatrix} 0 \\ 2 \end{bmatrix} - \sin 2t \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right) + c_2 e^t \left(\sin 2t \begin{bmatrix} 0 \\ 2 \end{bmatrix} + \cos 2t \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right)$$

Applying the initial condition yields

$$\vec{\mathbf{x}}(0) = \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -c_2 \\ 2c_1 \end{bmatrix} \implies \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

so that the solution to the initial value problem is

$$\vec{\mathbf{x}}(t) = e^t \begin{bmatrix} \sin 2t - \cos 2t \\ 2\cos 2t + 2\sin 2t \end{bmatrix}$$

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3. [2360/072823 (20 pts)] Consider the system of differential equations $\vec{\mathbf{x}}' = \begin{bmatrix} 1 & -a \\ 2 & -2 \end{bmatrix} \vec{\mathbf{x}}$.

   (a) (16 pts) Find all values of a, if any, such that the isolated fixed point (equilibrium solution) at the origin is

i. a center
ii. a stable node
iii. a saddle point
iv. an unstable spiral

(b) (4 pts) Find all values of a, if any, such that the system has infinitely many equilibrium solutions (fixed points).

SOLUTION:

$$\text{Tr}\,\mathbf{A} = -1 \quad \text{and} \quad |\mathbf{A}| = 2(a-1) \quad \text{and} \quad (\text{Tr}\,\mathbf{A})^2 - 4|\mathbf{A}| = 9 - 8a$$

(a) i. We need $\text{Tr}\,\mathbf{A} = 0$, which is not possible since $\text{Tr}\,\mathbf{A} = -1$, independent of a. No values of a will work.

ii. We need $(\text{Tr}\,\mathbf{A})^2 - 4|\mathbf{A}| > 0$, $|\mathbf{A}| > 0$ and $\text{Tr}\,\mathbf{A} < 0$. The first requires $9 - 8a > 0 \implies 9/8 > a$, the second requires $a > 1$ and the third does not depend on a. Thus $1 < a < 9/8$

iii. We need $|\mathbf{A}| < 0$, which happens when $a < 1$.

iv. We need $(\text{Tr}\,\mathbf{A})^2 - 4|\mathbf{A}| < 0$ and $\text{Tr}\,\mathbf{A} > 0$. The second inequality is impossible. No values of a will work.

(b) We need $|\mathbf{A}| = 0$ which requires $a = 1$.

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4. [2360/072823 (22 pts)] Consider the piecewise defined function $f(t) = \begin{cases} 0 & 0 \le t < 1 \\ e^t & 1 \le t < 2 \\ 0 & 2 \le t \end{cases}$

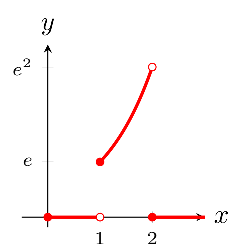
(a) (6 pts) Graph the function. Be sure to label important points on the axes.

(b) (8 pts) Write the function using step functions.

(c) (8 pts) Find the Laplace transform of the function.

SOLUTION:

(a) Graph.



(b) $f(t) = e^t\,[\text{step}(t-1) - \text{step}(t-2)]$

(c) $\mathscr{L}\{f(t)\} = e^{-s}\mathscr{L}\left\{e^{t+1}\right\} - e^{-2s}\mathscr{L}\left\{e^{t+2}\right\} = e^{-s}e\mathscr{L}\left\{e^t\right\} - e^{-2s}e^2\mathscr{L}\left\{e^t\right\} = \dfrac{e^{-s}e}{s-1} - \dfrac{e^{-2s}e^2}{s-1} = \dfrac{e^{1-s} - e^{2-2s}}{s-1}$

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5. [2360/072823 (30 pts)] Consider a damped harmonic oscillator governed by the linear operator $L(\vec{\mathbf{x}}) = \ddot{x} + b\dot{x} + 20x$.

(a) (5 pts) What is required for the solution to $L(\vec{\mathbf{x}}) = -2\cos\left(2\sqrt{5}t\right)$ to be unbounded? Explain briefly.

(b) (5 pts) What value(s) of b, if any, will allow solutions to the differential equation to pass through the t-axis at most once?

(c) (20 pts) With $b = 4$, solve the initial value problem $L(\vec{\mathbf{x}}) = \delta\left(t - \frac{\pi}{2}\right)$, $x(0) = 1$, $\dot{x}(0) = -2$. Note $\sin(x-y) = \sin x \cos y - \cos x \sin y$.

SOLUTION:

(a) The circular frequency $\omega_0 = \sqrt{20} = 2\sqrt{5}$ so solutions will be unbounded if the oscillator is undamped $(b = 0)$ since the forcing frequency matches the circular frequency.

(b) The oscillator must be critically or overdamped. $b^2 - 4mk \ge 0 \implies b^2 \ge 4(1)(20) \implies |b| \ge \sqrt{(4)(4)(5)} \implies b \ge 4\sqrt{5}$. Remark: The case $b \le -4\sqrt{5}$ is not considered since b must be nonnegative in the oscillator equation.

(c) Use Laplace Transforms.

$$\mathscr{L}\{\ddot{x} + 4\dot{x} + 20x\} = \mathscr{L}\left\{\delta\left(t - \frac{\pi}{2}\right)\right\}$$

$$s^2X(s) - sx(0) - \dot{x}(0) + 4\left[sX(s) - x(0)\right] + 20X(s) = e^{-\pi s/2}$$

$$\left(s^2 + 4s + 20\right)X(s) = e^{-\pi s/2} + sx(0) + \dot{x}(0) + 4x(0) = e^{-\pi s/2} + s + 2$$

$$X(s) = \frac{e^{-\pi s/2} + s + 2}{s^2 + 4s + 20} = \frac{e^{-\pi s/2}}{4}\left[\frac{4}{(s+2)^2 + 16}\right] + \frac{s+2}{(s+2)^2 + 16}$$

$$x(t) = \mathscr{L}^{-1}\left\{X(s)\right\} = \frac{1}{4}e^{-2(t-\pi/2)}\sin 4\left(t - \frac{\pi}{2}\right)\text{step}\left(t - \frac{\pi}{2}\right) + e^{-2t}\cos 4t$$

$$x(t) = e^{-2t}\left[\frac{e^{\pi}}{4}(\sin 4t)\ \text{step}\left(t - \frac{\pi}{2}\right) + \cos 4t\right]$$

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6. [2360/072823 (23 pts)] A 500-gallon tank initial contains 50 pounds of salt dissolved in 100 gallons of water. A brine (salt) solution containing 2 pounds of salt per gallon enters the tank at a rate of 20 gallons per minute. The well-mixed solution in the first tank empties into a second 500-gallon tank, which is initially empty, at a rate 20 gallon per minute. The second tank is also being filled with fresh water at a rate of 5 gallons per minute. The well-mixed solution in the second tank drains at a rate of 20 gallons per minute.

(a) (4 pts) Find the volumes, $V_1(t)$ and $V_2(t)$, of salt solution in each tank at time t.

(b) (15 pts) Set up, but **do not solve**, the initial value problem describing the amount of salt in each tank as a function of time. Write your final answer using matrices and vectors.

(c) (4 pts) Over what time interval is the system of differential equations valid?

SOLUTION:

(a) For the first tank, the flow in and out are equal, meaning the volume is constant, $V_1(t) = 100$. The second tank has a net flow in of 5 gallons per minute and starts out empty, meaning that the tank's volume is $V_2(t) = 5t$.

(b) Tank 1.

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= \text{mass flow in} - \text{mass flow out}\\ &= 20(2) - \frac{x_1}{V_1}(20)\\ &= 40 - \frac{x_1}{5}\end{aligned}$$

with the initial condition $x(0) = 50$.

Tank 2:

$$\begin{aligned}\frac{\mathrm{d}x_2}{\mathrm{d}t} &= \text{mass flow in} - \text{mass flow out}\\ &= \frac{x_1}{V_1}(20) + (5)(0) - \frac{x_2}{V_2}(20)\\ &= \frac{x_1}{5} - \frac{4}{t}x_2\end{aligned}$$

with the initial condition that $x_2(0) = 0$.

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= 40 - \frac{x_1}{5}, \quad x_1(0) = 50\\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= \frac{x_1}{5} - \frac{4x_2}{t}, \quad x_2(0) = 0\end{aligned}$$

Using matrices and vectors we have

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{5} & 0 \\ \frac{1}{5} & -\frac{4}{t} \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 40 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 50 \\ 0 \end{bmatrix}$$

(c) Since only the volume of fluid in Tank 2 is increasing, the equations are valid until Tank 2 is full. This happens when $500 = 5t$ which gives us $t = 100$ minutes. Therefore, the initial value problem in part (b) is valid on the interval $0 \leq t \leq 100$.

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