- This exam is worth 150 points and has 6 problems.
- **Show all work and simplify your answers!** Answers with no justification will receive no points unless otherwise noted.
- **Please begin each problem on a new page.**
- **DO NOT** leave the exam until you have satisfactorily scanned and uploaded your exam to Gradescope.
- You are taking this exam in a proctored and honor code enforced environment. **NO** calculators, cell phones, or other electronic devices or the internet are permitted during the exam. You are allowed one 8.5"× 11" crib sheet with writing on two sides.
- Remote students are allowed use of a computer during the exam only for a live video of their hands and face and to view the exam in the Zoom meeting.

0. At the top of the first page that you will be scanning and uploading to Gradescope, write the following statement and sign your name to it: "I will abide by the CU Boulder Honor Code on this exam." **FAILURE TO INCLUDE THIS STATEMENT AND YOUR SIGNATURE MAY RESULT IN A PENALTY.**

1. [2360/072823 (25 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given.
   (a) The matrix $\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix}$ is in RREF.
   (b) The subset $\mathbb{W} = \left\{ (m,n) \in \mathbb{R}^2 \,\middle|\, m,n \text{ integers} \right\}$ is a subspace of $\mathbb{R}^2$.
   (c) For any matrix $\mathbf{B}$, $\left|\mathbf{B}\mathbf{B}^{\mathrm{T}}\right|$ is defined.
   (d) The linear system $\mathbf{C}\vec{\mathbf{y}} = \vec{\mathbf{d}}$, where $\mathbf{C}$ is an $n \times n$ matrix, has the unique solution $\vec{\mathbf{x}} = \mathbf{C}^{-1}\vec{\mathbf{d}}$ if and only if the RREF of $\mathbf{C} = \mathbf{I}$.
   (e) $e^{t^2/2}$ is the integrating factor for the differential equation $\dfrac{y'}{t^2} + ty = \sin t$.

2. [2360/072823 (30 pts)] Find the solution to the initial value problem $\vec{\mathbf{x}}' = \begin{bmatrix} 1 & 1 \\ -4 & 1 \end{bmatrix} \vec{\mathbf{x}}$, $\vec{\mathbf{x}}(0) = \begin{bmatrix} -1 & 2 \end{bmatrix}^{\mathrm{T}}$. Write your final answer as a single vector.

3. [2360/072823 (20 pts)] Consider the system of differential equations $\vec{\mathbf{x}}' = \begin{bmatrix} 1 & -a \\ 2 & -2 \end{bmatrix} \vec{\mathbf{x}}$.
   (a) (16 pts) Find all values of a, if any, such that the isolated fixed point (equilibrium solution) at the origin is
      i. a center
      ii. a stable node
      iii. a saddle point
      iv. an unstable spiral
   (b) (4 pts) Find all values of a, if any, such that the system has infinitely many equilibrium solutions (fixed points).

4. [2360/072823 (22 pts)] Consider the piecewise defined function $f(t) = \begin{cases} 0 & 0 \le t < 1 \\ e^t & 1 \le t < 2 \\ 0 & 2 \le t \end{cases}$
   (a) (6 pts) Graph the function. Be sure to label important points on the axes.
   (b) (8 pts) Write the function using step functions.
   (c) (8 pts) Find the Laplace transform of the function.

5. [2360/072823 (30 pts)] Consider a damped harmonic oscillator governed by the linear operator $L(\vec{\mathbf{x}}) = \ddot{x} + b\dot{x} + 20x$.
   (a) (5 pts) What is required for the solution to $L(\vec{\mathbf{x}}) = -2\cos\left(2\sqrt{5}t\right)$ to be unbounded? Explain briefly.
   (b) (5 pts) What value(s) of b, if any, will allow solutions to the differential equation to pass through the t-axis at most once?
   (c) (20 pts) With $b = 4$, solve the initial value problem $L(\vec{\mathbf{x}}) = \delta\left(t - \frac{\pi}{2}\right)$, $x(0) = 1$, $\dot{x}(0) = -2$. Note $\sin(x-y) = \sin x \cos y - \cos x \sin y$.

MORE PROBLEMS AND LAPLACE TRANSFORM TABLE ON REVERSE

6. [2360/072823 (23 pts)] A 500-gallon tank initial contains 50 pounds of salt dissolved in 100 gallons of water. A brine (salt) solution containing 2 pounds of salt per gallon enters the tank at a rate of 20 gallons per minute. The well-mixed solution in the first tank empties into a second 500-gallon tank, which is initially empty, at a rate 20 gallon per minute. The second tank is also being filled with fresh water at a rate of 5 gallons per minute. The well-mixed solution in the second tank drains at a rate of 20 gallons per minute.

(a) (4 pts) Find the volumes, $V_1(t)$ and $V_2(t)$, of salt solution in each tank at time t.

(b) (15 pts) Set up, but **do not solve**, the initial value problem describing the amount of salt in each tank as a function of time. Write your final answer using matrices and vectors.

(c) (4 pts) Over what time interval is the system of differential equations valid?

Short table of Laplace Transforms: $\mathscr{L}\{f(t)\} = F(s) \equiv \int_0^\infty e^{-st} f(t)\,\mathrm{d}t$

In this table, a, b, c are real numbers with $c \geq 0$, and $n = 0, 1, 2, 3, \ldots$

$$\mathscr{L}\left\{t^n e^{at}\right\} = \frac{n!}{(s-a)^{n+1}} \qquad \mathscr{L}\left\{e^{at}\cos bt\right\} = \frac{s-a}{(s-a)^2+b^2} \qquad \mathscr{L}\left\{e^{at}\sin bt\right\} = \frac{b}{(s-a)^2+b^2}$$

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{\mathrm{d}^n F(s)}{\mathrm{d}s^n} \qquad \mathscr{L}\left\{e^{at} f(t)\right\} = F(s-a) \qquad \mathscr{L}\{\delta(t-c)\} = e^{-cs}$$

$$\mathscr{L}\{tf'(t)\} = -F(s) - s\frac{\mathrm{d}F(s)}{\mathrm{d}s} \qquad \mathscr{L}\{f(t-c)\,\mathrm{step}(t-c)\} = e^{-cs}F(s) \qquad \mathscr{L}\{f(t)\,\mathrm{step}(t-c)\} = e^{-cs}\mathscr{L}\{f(t+c)\}$$

$$\mathscr{L}\left\{f^{(n)}(t)\right\} = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \cdots - f^{(n-1)}(0)$$