1. [2360/071423 (20 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given.

   (a) The equation of motion for a particular harmonic oscillator is $x(t) = \cos 2t - \sin 2t$. An equivalent expression is $x(t) = \sqrt{2}\cos\left(2t - \frac{\pi}{4}\right)$.

   (b) Any finite order, linear, homogeneous, constant coefficient differential equation that does not contain an undifferentiated term will always have a constant solution.

   (c) The differential equation $x'' + e^x = 2$ describes a conservative system.

   (d) The function $x(t) = e^{2t} + e^{-3t}$ is a possible solution to the differential equation for a harmonic oscillator governed by $mx'' + bx' + kx = 0$ with mass m, damping constant b and restoring constant k.

   (e) The solution space of the differential equation $y''' = 0$ is span $\left\{t^2 + t, t^2 + 2, 4t - 8\right\}$.

SOLUTION:

(a) **False** $x(t) = \sqrt{2}\cos\left(2t + \dfrac{\pi}{4}\right) = \sqrt{2}\cos\left(2t - \dfrac{7\pi}{4}\right)$ since $1 = \sqrt{2}\cos\delta$ and $-1 = \sqrt{2}\sin\delta$ imply that $\delta = -\frac{\pi}{4}$ or $\delta = \frac{7\pi}{4}$.

(b) **True** Each term in the characteristic equation will contain an r, implying that $r = 0$ is a root of the characteristic equation, further implying that $e^{0t} = 1$ is a solution of the differential equation.

(c) **True** The differential equation can be written as $mx'' + V'(x) = 0$ where $m = 1$, $V'(x) = e^x - 2$.

(d) **False** When $m, b, k > 0$, as they are in the equation describing an harmonic oscillator, the solutions must ultimately decay in time (or remain bounded in case the oscillator is undamped).

(e) **False** All three of the functions are solutions of the differential equation whose solution space has dimension 3. However, the functions are linearly dependent:

$$W(t) = \begin{vmatrix} t^2+t & t^2+2 & 4t-8 \\ 2t+1 & 2t & 4 \\ 2 & 2 & 0 \end{vmatrix} = 2(-1)^{3+1}\begin{vmatrix} t^2+2 & 4t-8 \\ 2t & 4 \end{vmatrix} + 2(-1)^{3+2}\begin{vmatrix} t^2+t & 4t-8 \\ 2t+1 & 4 \end{vmatrix}$$

$$= 2\left(4t^2 + 8 - 8t^2 + 16t\right) - 2\left(4t^2 + 4t - 8t^2 + 16t - 4t + 8\right)$$

$$= 2\left(-4t^2 + 16t + 8\right) - 2\left(-4t^2 + 16t + 8\right) = 0$$

and thus cannot be a basis for a three dimensional vector space. Consequently, their span is not the solution space of the differential equation.

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2. [2360/071423 (27 pts)] Let $\mathrm{L}(\vec{\mathbf{y}}) = 2t^2y'' + 5ty' + y$.

   (a) (8 pts) Find a basis for the solution space of $\mathrm{L}(\vec{\mathbf{y}}) = 0$ by assuming solutions of the form $y = t^r$.

   (b) (4 pts) Verify that your basis functions are indeed solutions to the equation.

   (c) (15 pts) Find the general solution of $\mathrm{L}(\vec{\mathbf{y}}) = 2t^4$ using variation of parameters.

SOLUTION:

(a) With $y = t^r, y' = rt^{r-1}, y'' = r(r-1)t^{r-2}$ we have

$$2t^2y'' + 5ty' + y = 2t^2[r(r-1)]t^{r-2} + 5t(rt^{r-1}) + t^r = t^r\left(2r^2 - 2r + 5r + 1\right) = t^r\left(2r^2 + 3r + 1\right) = 0$$

$$2r^2 + 3r + 1 = (2r+1)(r+1) = 0 \implies r = -1, -\frac{1}{2}$$

A basis for the solution space is thus $\left\{t^{-1/2}, t^{-1}\right\}$.

(b)

$$2t^2\left(t^{-1}\right)'' + 5t\left(t^{-1}\right)' + t^{-1} = 2t^2\left(2t^{-3}\right) + 5t\left(-t^{-2}\right) + t^{-1} = 4t^{-1} - 5t^{-1} + t^{-1} = 0$$

$$2t^2\left(t^{-1/2}\right)'' + 5t\left(t^{-1/2}\right)' + t^{-1/2} = 2t^2\left(\frac{3}{4}t^{-5/2}\right) + 5t\left(-\frac{1}{2}t^{-3/2}\right) + t^{-1/2} = \frac{3}{2}t^{-1/2} - \frac{5}{2}t^{-1/2} + t^{-1/2} = 0$$

(c) Rewrite the equation in the form $y'' + \frac{5}{2t}y' + \frac{y}{2t^2} = t^2$ so that $f(t) = t^2$ and let $y_1 = t^{-1}$ and $y_2 = t^{-1/2}$. We assume $y_p = v_1(t)y_1(t) + v_2(t)y_2(t)$.

$$W\left[y_1, y_2\right](t) = \begin{vmatrix} t^{-1} & t^{-1/2} \\ -t^{-2} & -\frac{1}{2}t^{-3/2} \end{vmatrix} = -\frac{1}{2}t^{-5/2} + t^{-5/2} = \frac{1}{2}t^{-5/2}$$

$$v_1(t) = \int \frac{-y_2 f(t)}{W[y_1, y_2](t)}\,\mathrm{dt} = \int \frac{-t^{-1/2}t^2}{\frac{1}{2}t^{-5/2}}\,\mathrm{dt} = -2\int t^4\,\mathrm{dt} = -\frac{2}{5}t^5$$

$$v_2(t) = \int \frac{y_1 f(t)}{W[y_1, y_2](t)}\,\mathrm{dt} = \int \frac{t^{-1}t^2}{\frac{1}{2}t^{-5/2}}\,\mathrm{dt} = 2\int t^{7/2}\,\mathrm{dt} = \frac{4}{9}t^{9/2}$$

$$y_p = \left(-\frac{2}{5}t^5\right)t^{-1} + \left(\frac{4}{9}t^{9/2}\right)t^{-1/2} = \frac{2}{45}t^4$$

The general solution is $y(t) = c_1 t^{-1} + c_2 t^{-1/2} + \frac{2}{45}t^4$.

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3. [2360/071423 (20 pts)] Characteristic equations for certain constant coefficient linear homogeneous differential equations are given, along with a forcing function, $f(t)$. Give the form of the particular solution you would use to solve the nonhomogeneous differential equations from which the characteristic equations were derived when using the Method of Undetermined Coefficients. **Do not** solve for the coefficients.

(a) $r(r-2)(r-1) = 0;\ f(t) = 2 + \sin t$

(b) $[r - (-2-2i)][r - (-2+2i)] = 0;\ f(t) = \cos 2t + t$

(c) $(r+4)(r-2) = 0;\ f(t) = e^{2t} + e^{4t}$

(d) $[r - (1-i)][r - (1+i)](r+3)^2 = 0;\ f(t) = e^t \cos t + e^{-t}\sin t + te^{-3t}$

(e) $r^3(r-1) = 0;\ f(t) = \cos 2t - \sin 3t + 1$

SOLUTION:

(a) $y_p = At + B\sin t + C\cos t$

(b) $y_p = At + B + C\cos 2t + D\sin 2t$

(c) $y_p = Ae^{4t} + Bte^{2t}$

(d) $y_p = te^t(A\cos t + B\sin t) + e^{-t}(C\cos t + D\sin t) + t^2(Et+F)e^{-3t}$

(e) $y_p = At^3 + B\cos 2t + C\sin 2t + D\cos 3t + E\sin 3t$

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4. [2360/071423 (33 pts)] A 2-kg mass is attached to spring with restoring/spring constant of 2 Nt/m. The apparatus is aligned horizontally with a damping constant of 5 Nt/m/sec, and is forced by $f(t) = 3e^{-t} + 4$ Nt. Initially, $x(0) = -4$ and $\dot{x}(0) = -3$.

(a) (2 pts) Where is the mass with respect to its equilibrium position when $t = 0$ and in what direction is it moving at that time?

(b) (3 pts) Is the oscillator over-, under-, or critically damped? Justify your answer.

(c) (3 pts) Is the oscillator in resonance? Justify your answer.

(d) (15 pts) Find the position of the mass at any time t, that is, solve an appropriate initial value problem.

(e) (2 pts) From your answer to part (d), identify the transient and steady state solutions.

(f) (8 pts) Write the initial value problem from part (d) as a system of differential equations/IVPs, using matrices and vectors in your answer.

SOLUTION:

(a) The mass is 4 m to the left of the equilibrium position and is moving to the left at 3 m/sec.

(b) $b^2 - 4mk = 5^2 - 4(2)(2) = 9 > 0 \implies$ overdamped

(c) The system is damped and it is not driven by a sinusoidal external force so cannot be in resonance.

(d) The differential equation is $2\ddot{x} + 5\dot{x} + 2x = 3e^{-t} + 4$. The characteristic equation for the associated homogeneous equation is

$$2r^2 + 5r + 2 = (2r+1)(r+2) = 0 \implies r = -\frac{1}{2}, 2 \implies x_h(t) = c_1 e^{-2t} + c_2 e^{-t/2}$$

We guess $x_p = Ae^{-t} + B$. Then $\dot{x}_p = -Ae^{-t}$ and $\ddot{x}_p = Ae^{-t}$ and

$$2\ddot{x}_p + 5\dot{x}_p + 2x_p = 2Ae^{-t} - 5Ae^{-t} + 2Ae^{-t} + 2B = -Ae^{-t} + 2B = 3e^{-t} + 4 \implies A = -3, B = 2$$

$$x_p = 2 - 3e^{-t}$$

The general solution is thus

$$x(t) = x_h(t) + x_p(t) = c_1 e^{-2t} + c_2 e^{-t/2} + 2 - 3e^{-t}$$

to which we apply the initial conditions, yielding

$$x(0) = c_1 + c_2 + 2 - 3 = -4 \implies c_1 + c_2 = -3$$
$$\dot{x}(0) = -2c_1 - \frac{1}{2}c_2 + 3 = -3 \implies -2c_1 - \frac{1}{2}c_2 = -6$$

$$c_1 = \frac{\begin{vmatrix} -3 & 1 \\ -6 & -\frac{1}{2} \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ -2 & -\frac{1}{2} \end{vmatrix}} = \frac{\frac{3}{2} + 6}{-\frac{1}{2} + 2} = \frac{\frac{15}{2}}{\frac{3}{2}} = 5 \qquad c_2 = \frac{\begin{vmatrix} 1 & -3 \\ -2 & -6 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ -2 & -\frac{1}{2} \end{vmatrix}} = \frac{-6 - 6}{-\frac{1}{2} + 2} = \frac{-12}{\frac{3}{2}} = -8$$

The solution to the initial value problem is $x(t) = 5e^{-2t} - 8e^{-t/2} - 3e^{-t} + 2$ which gives the position of the mass at any time t.

(e) Transient solution is $5e^{-2t} - 8e^{-t/2} - 3e^{-t}$ and steady state solution is 2.

(f) Rewrite the differential equation as $\ddot{x} = -\frac{5}{2}\dot{x} - x + \frac{3}{2}e^{-t} + 2$.

$$u_1 = x \qquad u_2 = \dot{x}$$

$$u_1' = \dot{x} = u_2 \qquad u_2' = \ddot{x} = -u_1 - \frac{5}{2}u_2 + \frac{3}{2}e^{-t} + 2$$

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -1 & -\frac{5}{2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{3}{2}e^{-t} + 2 \end{bmatrix}, \quad \begin{bmatrix} u_1(0) \\ u_2(0) \end{bmatrix} = \begin{bmatrix} -4 \\ -3 \end{bmatrix}$$

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