1. [2360/063023 (24 pts)] The following parts (a) and (b) are not related.

   (a) (12 pts) Suppose that $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are $n \times n$ matrices with $|\mathbf{A}| = 0$, $|\mathbf{B}| = 3$, $|\mathbf{C}| = 1$ and that $\mathbf{D}$ is an $m \times n$ matrix. Calculate the following or explain why they fail to exist.

   i. $|\mathbf{CB}|$
   ii. $|\mathbf{C}^{\mathrm{T}}|$
   iii. $\left|\mathbf{C}^2\left(\mathbf{A}^{\mathrm{T}}\right)^{-1}\right|$
   iv. $|\mathbf{DA}|$

   (b) (12 pts) Let $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 1 & 3 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$. Calculate the following or explain why they fail to exist.

   i. $\mathbf{AB}$
   ii. $\mathbf{AB}^{\mathrm{T}}$
   iii. $\mathbf{A} + \mathbf{B}$
   iv. $\left(\mathbf{A}^{\mathrm{T}}\mathbf{A}\right)^{\mathrm{T}}$

SOLUTION:

(a) i. $|\mathbf{CB}| = |\mathbf{C}||\mathbf{B}| = (1)(3) = 3$
   ii. $\left|\mathbf{C}^{\mathrm{T}}\right| = |\mathbf{C}| = 1$
   iii. does not exist, since $|\mathbf{A}| = 0$
   iv. does not exist, because $\mathbf{DA}$ is not square

(b) i. Does not exist because the dimensions are a 2×3 and 2×3

   ii. $\mathbf{AB}^{\mathrm{T}} = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 4 & 7 \end{bmatrix}$

   iii. $\mathbf{A} + \mathbf{B} = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 1 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 2 & 5 \end{bmatrix}$

   iv. $\left(\mathbf{A}^{\mathrm{T}}\mathbf{A}\right)^{\mathrm{T}} = \left[\begin{bmatrix} 0 & 2 \\ 1 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 2 & 1 & 3 \end{bmatrix}\right]^{\mathrm{T}} = \begin{bmatrix} 4 & 2 & 6 \\ 2 & 2 & 3 \\ 6 & 3 & 9 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 4 & 2 & 6 \\ 2 & 2 & 3 \\ 6 & 3 & 9 \end{bmatrix}$

   Alternatively,

$$\left(\mathbf{A}^{\mathrm{T}}\mathbf{A}\right)^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}}\left(\mathbf{A}^{\mathrm{T}}\right)^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}}\mathbf{A} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 6 \\ 2 & 2 & 3 \\ 6 & 3 & 9 \end{bmatrix}$$

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2. [2360/063023 (16 pts)] The following parts (a) and (b) are not related.

   (a) (12 pts) Decide if the following subsets, $\mathbb{W}$, of the given vector space, $\mathbb{V}$, are subspaces. Assume the standard operations of vector addition and scalar multiplication apply. Justify the correct answer completely for full credit. A simple yes/no will result in zero points.

   i. $\mathbb{V} = \mathcal{C}([0,1])$; $\mathbb{W} = \left\{ f(t) \in \mathcal{C}([0,1]) \;\middle|\; \int_0^1 f(t)\,\mathrm{d}t = 0 \right\}$
   ii. $\mathbb{V} = \mathbb{M}_{22}$; $\mathbb{W} = \left\{ \begin{bmatrix} 1 & a \\ b & 1 \end{bmatrix} \;\middle|\; a, b \in \mathbb{R} \right\}$
   iii. $\mathbb{V} = \mathbb{R}^3$; $\mathbb{W} = \left\{ \begin{bmatrix} a \\ 0 \\ b \end{bmatrix} \;\middle|\; a, b \in \mathbb{R} \right\}$

   (b) (4 pts) Determine whether or not the set $\mathbb{S} = \{4, t - t^2, t^3\}$ forms a basis for some vector space. If so, what is the vector space's dimension?

SOLUTION:

(a) i. $\mathbb{W}$ is a subspace of $\mathbb{V}$. For f_1 and f_2 in $\mathbb{W}$ and real numbers a, b

$$\int_0^1 [af_1(t) + bf_2(t)]\,\mathrm{d}t = \int_0^1 af_1(t)\,\mathrm{d}t + \int_0^1 bf_2(t)\,\mathrm{d}t = a\int_0^1 f_1(t)\,\mathrm{d}t + b\int_0^1 f_2(t)\,\mathrm{d}t$$
$$= (a)(0) + (b)(0) = 0 \implies af_1(t) + bf_2(t) \in \mathbb{W}$$

ii. $\mathbb{W}$ is not a subspace of $\mathbb{V}$ because the zero vector, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, is not included in $\mathbb{W}$. It is also not closed under vector addition or scalar multiplication.

iii. $\mathbb{W}$ is a subspace of $\mathbb{V}$. For $\vec{\mathbf{v}}_1 = \begin{bmatrix} a_1 \\ 0 \\ b_1 \end{bmatrix}$ and $\vec{\mathbf{v}}_2 = \begin{bmatrix} a_2 \\ 0 \\ b_2 \end{bmatrix}$ and real numbers c_1 and c_2 we have

$$c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 = c_1\begin{bmatrix} a_1 \\ 0 \\ b_1 \end{bmatrix} + c_2\begin{bmatrix} a_2 \\ 0 \\ b_2 \end{bmatrix} = \begin{bmatrix} c_1a_1 + c_2a_2 \\ 0 \\ c_1b_1 + c_2b_2 \end{bmatrix} \in \mathbb{W}$$

(b) Since we have not specified a specific vector space that these may be basis for, we need only show that they are linearly independent. They then are a basis for their own span. The simplest way to do this is to apply the Wronskian test.

$$W\left[4, t - t^2, t^3\right](t) = \begin{vmatrix} 4 & t - t^2 & t^3 \\ 0 & 1 - 2t & 3t^2 \\ 0 & -2 & 6t \end{vmatrix} = 4\begin{vmatrix} 1 - 2t & 3t^2 \\ -2 & 6t \end{vmatrix}$$
$$= 4\left[(1 - 2t)6t + 6t^2\right]$$
$$= 4\left(6t - 6t^2\right) = 24t(1 - t) \neq 0 \text{ for all } t \neq 0, 1$$

Therefore, the vectors are linearly independent and are a basis for their own span, which is a vector space of dimension 3.

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3. [2360/063023 (19 pts)] Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

(a) (6 pts) Show that $\lambda = 0$ and $\lambda = 2$ are eigenvalues of the matrix.

(b) (4 pts) State the algebraic multiplicity for the eigenvalues in part (a).

(c) (6 pts) Find the eigenspace associated with $\lambda = 0$ and state its dimension.

(d) (3 pts) Is it possible that the system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$, where $\vec{\mathbf{b}} \neq \vec{\mathbf{0}}$, could be inconsistent? Explain briefly.

SOLUTION:

(a)

$$\begin{vmatrix} 1 - \lambda & 1 & 0 \\ 1 & 1 - \lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = -\lambda\begin{vmatrix} 1 - \lambda & 1 \\ 1 & 1 - \lambda \end{vmatrix} = -\lambda\left[(1 - \lambda)^2 - 1\right]$$
$$= -\lambda\left(\lambda^2 - 2\lambda\right)$$
$$= -\lambda^2(\lambda - 2) = 0$$

Therefore, $\lambda = 0$ and $\lambda = 2$ are eigenvalues of $\mathbf{A}$.

(b) The eigenvalue $\lambda = 0$ has algebraic multiplicity 2 and the eigenvalue $\lambda = 2$ has algebraic multiplicity 1.

(c)

$$(\mathbf{A} - 0\mathbf{I})\vec{\mathbf{v}} = \vec{\mathbf{0}}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\vec{\mathbf{v}} = \begin{bmatrix} -r \\ r \\ s \end{bmatrix}$$

$$= r \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad r, s \in \mathbb{R}$$

Therefore, the eigenspace for eigenvalue $\lambda = 0$ is

$$\mathbb{E}_{\lambda=0} = \text{span} \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

having dimension 2.

(d) Yes, since $\lambda = 0$ is an eigenvalue, $|\mathbf{A}| = 0$, meaning that $\mathbf{A}$ is singular. Therefore, the linear system will either have infinitely many solutions, or no solution at all (be inconsistent).

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4. [2360/063023 (21 pts)] The following problems are not related.

(a) (6 pts) Find the RREF of the matrix $\mathbf{A} = \begin{bmatrix} 3 & 0 & 6 \\ 1 & 3 & 2 \\ 2 & 6 & 4 \end{bmatrix}$.

(b) (15 pts) The following is the augmented matrix from a system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$.

$$\left[\begin{array}{ccccc|c} 0 & 1 & 0 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

i. (3 pts) Is the matrix in RREF? If so, write YES. If not, put the matrix in RREF.

ii. (6 pts) Find a particular solution to the nonhomogeneous system.

iii. (6 pts) Find a basis for the solution space of the associated homogeneous problem. What is the dimension of this solution space?

SOLUTION:

(a)

$$\begin{bmatrix} 3 & 0 & 6 \\ 1 & 3 & 2 \\ 2 & 6 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 1 & 3 & 2 \\ 2 & 6 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \\ 0 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(b) No. The RREF is $\left[\begin{array}{ccccc|c} 0 & 1 & 0 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$.

(c) The pivot columns correspond to x_2, x_3 and x_5 (leading variables) with nonpivot columns corresponding to x_1 and x_4 (free variables), which we set to r and s, respectively. This gives

$$\begin{aligned} x_1 &= r \quad \text{free variable} \\ x_2 + 2x_4 &= 3 \\ x_3 + x_4 &= 2 \\ x_4 &= s \quad \text{free variable} \\ x_5 &= 1 \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} r \\ 3-2s \\ 2-s \\ s \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 2 \\ 0 \\ 1 \end{bmatrix} + r \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ -2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \quad r, s \in \mathbb{R}$$

i. A particular solution to the nonhomogeneous system is, setting $r = s = 0$, $\vec{\mathbf{x}}_p = \begin{bmatrix} 0 \\ 3 \\ 2 \\ 0 \\ 1 \end{bmatrix}$.

ii. A basis for the solution space of the corresponding homogeneous system is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

The dimension of the solution space is 2.

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5. [2360/063023 (20 pts)] Let $\vec{\mathbf{v}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\vec{\mathbf{b}} = \begin{bmatrix} -3 \\ 6 \end{bmatrix}$.

(a) (5 pts) Compute $\vec{\mathbf{v}}^{\mathrm{T}}\vec{\mathbf{v}}$ and $\mathrm{Tr}\left(\vec{\mathbf{v}}^{\mathrm{T}}\vec{\mathbf{v}}\right)$.

(b) (5 pts) Can Cramer's Rule be used to solve the linear system $\left(\vec{\mathbf{v}}\vec{\mathbf{v}}^{\mathrm{T}}\right)\vec{\mathbf{x}} = \vec{\mathbf{b}}$? Justify your answer.

(c) (10 pts) Compute $\mathbf{H} = \mathbf{I} - 2\vec{\mathbf{v}}\vec{\mathbf{v}}^{\mathrm{T}}$ and use its inverse to solve $\mathbf{H}\vec{\mathbf{x}} = \vec{\mathbf{b}}$. Perform Gauss-Jordan elimination to find the inverse.

SOLUTION:

(a)

$$\vec{\mathbf{v}}^{\mathrm{T}}\vec{\mathbf{v}} = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \end{bmatrix}$$

$$\mathrm{Tr}\left(\vec{\mathbf{v}}^{\mathrm{T}}\vec{\mathbf{v}}\right) = \mathrm{Tr}\begin{bmatrix} 2 \end{bmatrix} = 2$$

(b) No, $\vec{\mathbf{v}}\vec{\mathbf{v}}^{\mathrm{T}}$ is singular.

$$\vec{\mathbf{v}}\vec{\mathbf{v}}^{\mathrm{T}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \implies \left|\vec{\mathbf{v}}\vec{\mathbf{v}}^{\mathrm{T}}\right| = \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} = 0$$

(c)

$$\mathbf{H} = \mathbf{I} - 2\vec{\mathbf{v}}\vec{\mathbf{v}}^{\mathrm{T}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} -1 & 2 & 1 & 0 \\ 2 & -1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} -1 & 2 & 1 & 0 \\ 0 & 3 & 2 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} -1 & 2 & 1 & 0 \\ 0 & 1 & \frac{2}{3} & \frac{1}{3} \end{array}\right] \sim \left[\begin{array}{cc|cc} -1 & 0 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 1 & \frac{2}{3} & \frac{1}{3} \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 1 & \frac{2}{3} & \frac{1}{3} \end{array}\right]$$

$$\mathbf{H}^{-1} = \tfrac{1}{3}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

$$\vec{\mathbf{x}} = \mathbf{H}^{-1}\vec{\mathbf{b}} = \tfrac{1}{3}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -3 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

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