1. [2360/061623 (18 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown. No partial credit given.

   (a) The operator $\mathrm{L}(y) = [\ln(\sin t)]y' + 3y$ is a linear operator.

   (b) Let $y' = f(y)$, where $f$ is continuous. An $f(y)$ exists such that there are only two equilibrium solutions, both of which are semistable.

   (c) The equation $(y')^2 + (\sin t)y + 1 = 0$ is linear, second order, homogeneous, autonomous differential equation.

   (d) The differential equation $(\cos t)y' + (\sin t)y = t$ can be solved using integrating factor method.

   (e) Picard's theorem guarantees the existence of a unique solution to the initial problem: $y' = (\ln t)y^{1/2}$, $y(1) = 0$.

   (f) The differential equation describing Newton's Law of Cooling possesses a single, stable equilibrium solution.

**SOLUTION:**

(a) **TRUE** $\mathrm{L}(k\vec{\mathbf{y}}) = k\mathrm{L}\vec{(\mathbf{y}})$ and $\mathrm{L}(\vec{\mathbf{x}} + \vec{\mathbf{y}}) = \mathrm{L}(\vec{\mathbf{x}}) + \mathrm{L}\mathbf{y})$ where $k \in \mathbb{R}$.

(b) **TRUE** One example is $f(y) = \left(y^2 - 1\right)^2$

(c) **FALSE** The given equation is nonlinear due to the $(y')^2$ term.

(d) **TRUE** This is a linear, first order ODE and therefore can be solved using integrating factor method.

(e) **FALSE** The derivative $f_y(t, y) = \dfrac{\ln(t)}{2y^{1/2}}$ is undefined at the initial condition. Therefore Picard's theorem tells us nothing about the uniqueness of the solution to the ODE.

(f) **TRUE** $T'(t) = k(M - T)$, $k > 0$ has the single equilibrium solution $T = M$. For $T_0 < M, T'(t) > 0$ and for $T_0 > M, T'(t) < 0$ implying that the equilibrium solution is stable.

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2. [2360/061623 (20 pts)] The following questions are unrelated.

   (a) (14 pts) Solve the following initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{e^{-y}}{2t - 1},\ y(0) = 0.$$

   (b) (6 pts) Consider the following initial value problem

$$y' = (\ln t)y,\ y(e) = 1.$$

   Apply two steps of Euler's method using a step size $h = e$.

**SOLUTION:**

(a) Note that this differential equation is separable and has no equilibrium solutions. Then

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{e^{-y}}{2t - 1}$$

$$\int e^y\,\mathrm{d}y = \int \frac{1}{2t - 1}\,\mathrm{d}t$$

$$e^y = \frac{1}{2}\int \frac{1}{u}\,\mathrm{d}u \quad (\text{where } u = 2t - 1)$$

$$e^y = \frac{1}{2}\ln|2t - 1| + C$$

$$y = \ln\left(\frac{1}{2}\ln|2t - 1| + C\right)$$

Applying the initial condition gives

$$0 = \ln\left(\frac{1}{2}\ln|2(0)-1| + C\right)$$
$$0 = \ln(C)$$
$$C = 1$$

and finally, $y(t) = \ln\left(\frac{1}{2}\ln|2t-1| + 1\right)$.

(b) The iteration for Euler's method is: $y_{n+1} = y_n + hf(t_n, y_n) = y_n + e\ln(t_n)y_n$. Therefore, we have

$$y_0 = 1$$
$$y_1 = 1 + e\ln(e)(1) = 1 + e$$
$$y_2 = (1+e) + e\ln(2e)(1+e) = (1+e)[1 + e\ln(2e)] = (1+e)[1 + e(\ln 2 + 1)] = (1+e)(1 + e + e\ln 2)$$

■

3. [2360/061623 (20 pts)] Consider the following system of differential equations:

$$x' = x^2 + y^2 - 4$$
$$y' = y - xy$$

(a) (5 pts) Find the $h$ nullclines

(b) (5 pts) Find the $v$ nullclines

(c) (6 pts) On a single phase plane, plot the $h$ nullclines as solid curves/lines and the $v$ nullclines as dashed curves/lines.

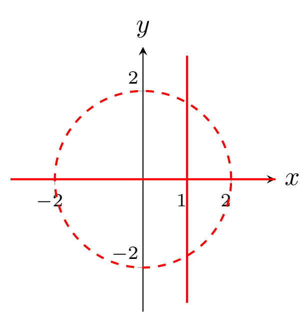
(d) (4 pts) Find all equilibrium points, provided any exist.

**Solution:**

(a) The $h$ nullclines occur when $y' = 0$. Therefore, these come from $0 = y - xy = y(1-x)$ which gives $y = 0$ and $x = 1$.

(b) The $v$ nullclines occur when $x' = 0$. Therefore, these come from $0 = x^2 + y^2 - 4$ which gives the circle $x^2 + y^2 = 4$.

(c) Sketch of nullclines.




(d) Equilibrium solutions occur where the $h$ and $v$ nullclines intersect. Since the lines $y = 0$ and $x = 1$ both cross through the circle $x^2 + y^2 = 4$ there are multiple equilibria. First, when $y = 0$ we have

$$x^2 = 4$$
$$x = \pm 2$$

Therefore, $(2, 0)$ and $(-2, 0)$ are equilibrium points. The other two occur when $x = 1$ intersects the circle. This gives

$$1 + y^2 = 4$$
$$y^2 = 3$$
$$y = \pm\sqrt{3}$$

Therefore, $(1, \sqrt{3})$ and $(1, -\sqrt{3})$ are also equilibrium points.

■

4. [2360/061623 (18 pts)] A tank with a total capacity of 4600 liters (L) initially contains 600 L of fresh water. Water containing $\dfrac{\ln(t+1)}{(t+1)^2}$ kilograms (kg) of sugar per liter flows into the tank at a rate of 6 L/min, and the well mixed sugar solution is allowed to drain from the bottom of the tank at a rate of 2 L/min. Let $x(t)$ denote the amount (kg) of sugar in the tank at time $t$.

(a) (14 pts) Write down, but **do not solve**, the initial value problem that $x(t)$ satisfies.

(b) (4 pts) Without solving the initial value problem from part (a), state the interval over which the solution will be valid based on the physical situation.

**SOLUTION:**

(a) First, determine the rate of change of volume of liquid in the tank.

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t} &= \text{flow in} - \text{flow out} \\ &= 6\,\frac{\text{L}}{\text{min}} - 2\,\frac{\text{L}}{\text{min}} \\ &= 4\,\frac{\text{L}}{\text{min}}\end{aligned}$$

Since initially there are 600 L in the tank, the amount of volume of liquid in the tank at time $t$ is

$$V(t) = 4t + 600 \;\text{ liters}$$

Next, the rate of change of the amount of sugar in the tank is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \text{mass flow in} - \text{mass flow out}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \left(\frac{\ln(t+1)}{(t+1)^2}\,\frac{\text{kg}}{\text{L}}\right)\left(6\,\frac{\text{L}}{\text{min}}\right) - \left(\frac{x}{4t+600}\,\frac{\text{kg}}{\text{L}}\right)\left(2\,\frac{\text{L}}{\text{min}}\right)$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{x}{2t+300} = \frac{6\ln(t+1)}{(t+1)^2}$$

Together with the initial condition that the tank contains no sugar, yields the initial value problem

$$\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{x}{2t+300} = \frac{6\ln(t+1)}{(t+1)^2}, \quad x(0) = 0$$

(b) Since the net flow is into the tank, the initial value problem is valid until the volume in the tank is 4600 L (completely full).

$$\begin{aligned}4600 &= 4t + 600 \\ t &= 1000\end{aligned}$$

Therefore, the interval on which the initial value problem in part (a) is valid is $0 \le t \le 1000$ min.

■

5. [2360/061623 (24 pts)] Consider the differential equation $ty' = y + \frac{1}{t}$.

(a) (4 pts) Show that $y_h = Ct$ is the general solution of the associated homogeneous equation, noting that $C$ is an arbitrary constant.

(b) (15 pts) Use variation of parameters (Euler-Lagrange two-stage method) to find a particular solution to the original nonhomogeneous equation. Show all your work.

(c) (5 pts) Find the solution to the differential equation passing through the point $\left(2, \frac{3}{4}\right)$.

**SOLUTION:**

(a) This can be done by substituting this into the corresponding homogeneous equation.

$$\begin{aligned}ty_h' &= y_h \\ t(Ct)' &= Ct \\ Ct &= Ct \quad \checkmark\end{aligned}$$

Therefore, $y_h = Ct$ is a solution to the corresponding homogeneous equation.

(b) Step one is already complete, $y_h = Ct$ is a solution to the corresponding homogeneous equation. Replacing $C$ with $v(t)$ and substituting this into the standard form of the equation $y' = \frac{y}{t} + \frac{1}{t^2}$ gives

$$\begin{aligned}
[v(t)t]' &= v(t) + \frac{1}{t^2} \\
[v'(t)t + v(t)] &= v(t) + \frac{1}{t^2} \\
v'(t) &= \frac{1}{t^3} \\
v(t) &= \int \frac{1}{t^3}\,\mathrm{d}t \\
&= -\frac{1}{2t^2}
\end{aligned}$$

Therefore, a particular solution to the original nonhomogeneous ODE is $y_p = -\frac{1}{2t^2}t = -\frac{1}{2t}$.

(c) By the Nonhomogeneous Principle, the general solution to the original nonhomogeneous ODE is

$$y(t) = y_h(t) + y_p(t) = Ct - \frac{1}{2t}$$

Using the given point (applying the initial condition) yields $\frac{3}{4} = 2C - \frac{1}{4} \implies C = \frac{1}{2}$ so the solution is

$$y(t) = \frac{1}{2}\left(t - \frac{1}{t}\right)$$

■