1. [2360/072222 (30 pts)] Solve the initial value problem $\ddot{x} + 2\dot{x} + 10x = \delta(t-3),\ x(0) = 0,\ \dot{x}(0) = 0.$

SOLUTION:

$$s^2X(s) - sx(0) - x'(0) + 2[sX(s) - x(0)] + 10X(s) = e^{-3s}$$

$$\left(s^2 + 2s + 10\right) X(s) = e^{-3s} \quad \text{(complete the square)}$$

$$X(s) = \frac{e^{-3s}}{(s+1)^2 + 9} = \frac{1}{3}\left(\frac{3}{(s+1)^2 + 3^2}\right) e^{-3s}$$

$$x(t) = \mathscr{L}^{-1}\left\{\frac{1}{3}\left(\frac{3}{(s+1)^2 + 3^2}\right) e^{-3s}\right\} = \frac{1}{3} e^{-(t-3)} \sin[3(t-3)]\,\text{step}(t-3)$$

■

2. [2360/072222 (20 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown, no work will be graded and no partial credit will be given. Please arrange your answers in a 10×2 matrix with the letters (a)-(j) in column 1 and your answers to each part in the second column.

   (a) The parabola $y = x^2$ is a subspace of $\mathbb{R}^2$.

   (b) If $\mathbf{B}$ is an $m \times n$ matrix, then $\left|\mathbf{B}\mathbf{B}^{\mathrm{T}}\right|$ is defined.

   (c) The matrix $\mathbf{A} = \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ is in RREF and therefore the system from which this augmented matrix was derived has a unique solution.

   (d) The integrating factor for the equation $(t^2+1)y' + 2ty = \cos t$ is e^{t^2}.

   (e) The isoclines of $y' + y^2 = t$ are parabolas opening downward.

   (f) The following system has no equilibrium points:

$$x' = x^2 + y^2 + 1$$
$$y' = y^4 + \sqrt{x+1}$$

   (g) The equation $y' = y^2 - y$ has a stable equilibrium solution at $y = 1$.

   (h) There is only one value of b for which the harmonic oscillator governed by the differential equation $4\ddot{x} + b^2\dot{x} + 36x = \cos 3t$ will have unbounded solutions.

   (i) Consider the initial value problem $y' = f(t, y),\ y(1) = 1$ with $f(1,1) = 2$ and $f_y(1,1)$ not defined. Picard's theorem guarantees that the IVP does not have a unique solution.

   (j) Newton's Law of Cooling for a certain situation is given by $\dfrac{\mathrm{d}T}{\mathrm{d}t} = 2(50 - T)$. With the change of variable $y = 50 - T$, this is equivalent to an exponential decay problem.

SOLUTION:

   (a) **FALSE** The only subspaces of $\mathbb{R}^2$ are lines through the origin. More demonstrably, $(1,1)$ and $(2,4)$ are on the parabola but $(1,1) + (2,4) = (3,5)$ is not, showing that the set (the parabola) is not closed under vector addition.

   (b) **TRUE** $\mathbf{B}$ is $m \times n$ implying that $\mathbf{B}^{\mathrm{T}}$ is $n \times m$ further implying that $\mathbf{B}\mathbf{B}^{\mathrm{T}}$ is square $(m \times m)$ and thus $\left|\mathbf{B}\mathbf{B}^{\mathrm{T}}\right|$ is defined.

   (c) **FALSE** The matrix is in RREF but there is a nonpivot column so a solution exists but is not unique.

   (d) **FALSE** The integrating factor is $t^2 + 1$.

   (e) **FALSE** The isoclines are $t - y^2 = k$, parabolas that open to the right (axis of symmetry along the x-axis).

   (f) **TRUE** The system has no nullclines and thus no equilibrium points.

   (g) **FALSE** The equilibrium solution at $y = 1$ is unstable.

   (h) **TRUE** The oscillator must be undamped, so $b = 0$ only.

   (i) **FALSE** Since $f_y(1,1)$ is undefined, it cannot be continuous in a rectangle containing $(1,1)$. Thus no conclusions can be drawn from Picard's theorem.

(j) **TRUE**. The change of variable $y = 50 - T$ yields $y' = -2y$.

■

3. (35 pts) The following parts are not related.

   (a) (10 pts) Consider the initial value problem (IVP) $\quad ty' - 3(\ln t)^2 e^{-y} = 0, \quad y(1) = \ln 8.$

      i. (8 pts) Find the implicit solution to the IVP.

      ii. (2 pts) Find the explicit solution to the IVP and state the interval over which the solution is valid.

   (b) (25 pts) A particular solution to $L(\vec{\mathbf{y}}) = f$, where L is a linear operator, is $y_p = \cos t$. Suppose the characteristic equation for the associated homogeneous equation is $(r-2)(r^2-1) = 0$. Use Cramer's Rule to find the solution to the following initial value problem. No points for using other methods.

$$L(\vec{\mathbf{y}}) = f, \quad y(0) = 4,\ y'(0) = 0,\ y''(0) = -1$$

SOLUTION:

(a) i. The equation is separable.

$$\begin{aligned}
\int e^y \,\mathrm{d}y &= \int 3\frac{(\ln t)^2}{t}\,\mathrm{d}t \quad (u = \ln t)\\
e^y &= 3\int u^2\,\mathrm{d}u = (\ln t)^3 + C\\
e^{\ln 8} &= (\ln 1)^3 + C \implies C = 8\\
e^y &= (\ln t)^3 + 8
\end{aligned}$$

ii. The explicit solution is $y = \ln\left[(\ln t)^3 + 8\right]$. Clearly, $t > 0$ for input into the "inner" $\ln t$. For input into the "outer" natural logarithm function, we also need

$$(\ln t)^3 + 8 > 0 \implies \ln t > \sqrt[3]{-8} = -2 \implies t > e^{-2}$$

The solution is valid on $\left(e^{-2}, \infty\right)$.

(b) Based on the characteristic equation, $(r-2)(r+1)(r-1) = 0$, the solution to the homogeneous equation is $y_h = c_1 e^{2t} + c_2 e^{-t} + c_3 e^t$ so the general solution to which we apply the initial conditions is $y = y_h + y_p$.

$$\begin{aligned}
y(t) &= c_1 e^{2t} + c_2 e^{-t} + c_3 e^t + \cos t\\
y'(t) &= 2c_1 e^{2t} - c_2 e^{-t} + c_3 e^t - \sin t\\
y''(t) &= 4c_1 e^{2t} + c_2 e^{-t} + c_3 e^t - \cos t
\end{aligned}$$

At $t = 0$ we have

$$\begin{aligned}
c_1 + c_2 + c_3 + 1 &= 4\\
2c_1 - c_2 + c_3 &= 0\\
4c_1 + c_2 + c_3 - 1 &= -1
\end{aligned} \implies \begin{bmatrix} 1 & 1 & 1\\ 2 & -1 & 1\\ 4 & 1 & 1 \end{bmatrix}\begin{bmatrix} c_1\\ c_2\\ c_3 \end{bmatrix} = \begin{bmatrix} 3\\ 0\\ 0 \end{bmatrix}$$

Now use Cramer's Rule

$$\begin{vmatrix} 1 & 1 & 1\\ 2 & -1 & 1\\ 4 & 1 & 1 \end{vmatrix} = 1(-1)^{1+1}\begin{vmatrix} -1 & 1\\ 1 & 1 \end{vmatrix} + 1(-1)^{1+2}\begin{vmatrix} 2 & 1\\ 4 & 1 \end{vmatrix} + 1(-1)^{1+3}\begin{vmatrix} 2 & -1\\ 4 & 1 \end{vmatrix} = -2 + (-1)(-2) + 6 = 6$$

$$c_1 = \frac{\begin{vmatrix} 3 & 1 & 1\\ 0 & -1 & 1\\ 0 & 1 & 1 \end{vmatrix}}{6} = \frac{3(-1)^{1+1}\begin{vmatrix} -1 & 1\\ 1 & 1 \end{vmatrix}}{6} = \frac{3(-2)}{6} = -1$$

$$c_2 = \frac{\begin{vmatrix} 1 & 3 & 1\\ 2 & 0 & 1\\ 4 & 0 & 1 \end{vmatrix}}{6} = \frac{3(-1)^{1+2}\begin{vmatrix} 2 & 1\\ 4 & 1 \end{vmatrix}}{6} = \frac{-3(-2)}{6} = 1$$

$$c_3 = \frac{\begin{vmatrix} 1 & 1 & 3\\ 2 & -1 & 0\\ 4 & 1 & 0 \end{vmatrix}}{6} = \frac{3(-1)^{1+3}\begin{vmatrix} 2 & -1\\ 4 & 1 \end{vmatrix}}{6} = \frac{3(6)}{6} = 3$$

The solution to the initial value problem is thus

$$y(t) = -e^{2t} + e^{-t} + 3e^{t} + \cos t$$

4. [2360/072222 (29 pts)] The following parts are not related.

(a) (12 pts) Consider the function $f(t) = \begin{cases} 0 & t < 0 \\ t^2 & 0 \le t < 2 \\ 5 - t & 2 \le t \end{cases}$

i. (3 pts) Graph the function.

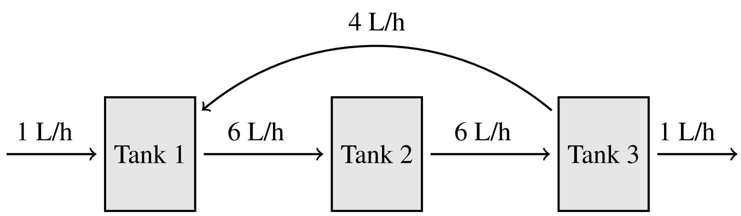
ii. (4 pts) Write the $f(t)$ as a single function using step functions.

iii. (5 pts) Find the Laplace transform of $f(t)$.

(b) (17 pts) Three 200 liter (L) tanks, $1, 2, 3$, contain solution that is always well-mixed. Initially, tanks 1 and 3 are half full of pure water and tank 2 is filled with water containing 10 grams (g) of dissolved sugar. Water with 2 g per liter (g/L) of dissolved sugar enters tank 1 at a rate of 1 liter per hour (L/h) and flows through the system as shown in the figure below.

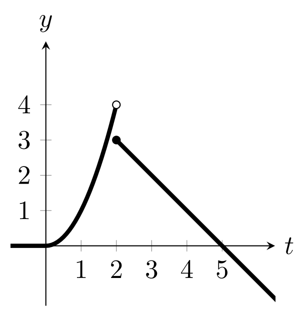
i. (15 pts) Set up, but do **NOT** solve, a system of differential equations that models the aforementioned scenario. Write your final answer using matrices and vectors.

ii. (2 pts) Over what interval of t will the solution be valid? Hint: You do not need to solve the system to answer this.



SOLUTION:

(a) i. Sketch.



ii.

$$\begin{aligned} f(t) &= t^2 \operatorname{step}(t) - t^2 \operatorname{step}(t-2) + (5-t)\operatorname{step}(t-2) \\ &= t^2 \left[\operatorname{step}(t) - \operatorname{step}(t-2)\right] + (5-t)\operatorname{step}(t-2) \\ &= t^2 \operatorname{step}(t) + (5 - t - t^2)\operatorname{step}(t-2) \end{aligned}$$

iii. Proceeding term-by-term:

$$\mathscr{L}\left\{t^2 \operatorname{step}(t)\right\} = e^{-0s}\mathscr{L}\left\{t^2\right\} = \frac{2}{s^3}$$

$$\mathscr{L}\left\{t^2 \operatorname{step}(t-2)\right\} = e^{-2s}\mathscr{L}\left\{(t+2)^2\right\} = e^{-2s}\mathscr{L}\left\{t^2+4t+4\right\} = e^{-2s}\left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right)$$

$$\begin{aligned} \mathscr{L}\left\{(5-t)\operatorname{step}(t-2)\right\} &= \mathscr{L}\left\{[3-(t-2)]\operatorname{step}(t-2)\right\} \\ &= \mathscr{L}\left\{3\operatorname{step}(t-2)\right\} - \mathscr{L}\left\{(t-2)\operatorname{step}(t-2)\right\} \\ &= \frac{3e^{-2s}}{s} - \frac{e^{-2s}}{s^2} = e^{-2s}\left(\frac{3}{s} - \frac{1}{s^2}\right) \end{aligned}$$

Thus

$$\mathscr{L}\left\{t^2\,\text{step}(t) - t^2\,\text{step}(t-2) + (5-t)\,\text{step}(t-2)\right\} = \frac{2}{s^3} - e^{-2s}\left(\frac{2}{s^3}+\frac{4}{s^2}+\frac{4}{s}\right) + e^{-2s}\left(\frac{3}{s}-\frac{1}{s^2}\right)$$
$$= \frac{2}{s^3} - e^{-2s}\left(\frac{2}{s^3}+\frac{5}{s^2}+\frac{1}{s}\right)$$

(b) i. Let $x_1(t), x_2(t), x_3(t)$ represent the mass (grams) of sugar and $V_1(t), V_2(t), V_3(t)$ the volume of solution (L) in Tank 1, 2, 3 at time t , respectively. Then with

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \text{flow rate in} - \text{flow rate out}$$

we have

$$\frac{\mathrm{d}V_1}{\mathrm{d}t} = 1+4-6 = -1 \quad V_1(0)=100 \implies \int \mathrm{d}V_1 = \int -1\,\mathrm{d}t \implies V_1(t) = 100-t$$
$$\frac{\mathrm{d}V_2}{\mathrm{d}t} = 6-6 = 0 \quad V_2(0)=200 \implies \int \mathrm{d}V_2 = \int 0\,\mathrm{d}t \implies V_2(t) = 200$$
$$\frac{\mathrm{d}V_3}{\mathrm{d}t} = 6-1-4 = 1 \quad V_3(0)=100 \implies \int \mathrm{d}V_3 = \int 1\,\mathrm{d}t \implies V_3(t) = 100+t$$

For the amount of sugar in each tank, we will use

$$\frac{\mathrm{d}x}{\mathrm{d}t} = (\text{flow rate in})(\text{concentration in}) - (\text{flow rate out})(\text{concentration out})$$

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = (1)(2) + 4\left(\frac{x_3}{100+t}\right) - 6\left(\frac{x_1}{100-t}\right)$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = 6\left(\frac{x_1}{100-t}\right) - 6\left(\frac{x_2}{200}\right)$$
$$\frac{\mathrm{d}x_3}{\mathrm{d}t} = 6\left(\frac{x_2}{200}\right) - 4\left(\frac{x_3}{100+t}\right) - 1\left(\frac{x_3}{100+t}\right)$$

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} \dfrac{-6}{100-t} & 0 & \dfrac{4}{100+t} \\ \dfrac{6}{100-t} & -\dfrac{3}{100} & 0 \\ 0 & \dfrac{3}{100} & \dfrac{-5}{100+t} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix}$$

ii. Tank 1 empties and Tank 3 fills after 100 hours so the interval over which the solution to the system is valid is $[0, 100]$.

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5. [2360/072222 (36 pts)] The following parts are not related.

(a) (24 pts) Solve the initial value problem $\vec{\mathbf{x}}' = \begin{bmatrix} -1 & 1 \\ -9 & 5 \end{bmatrix}\vec{\mathbf{x}}, \quad \vec{\mathbf{x}}(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$. Write your final answer as a single vector.

(b) (12 pts) Consider the linear system $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}}$ where $\mathbf{A} = \begin{bmatrix} k & 1 \\ -2 & k \end{bmatrix}$ and k is a real number.

i. (3 pts) For what values, if any, of k does the system have a unique equilibrium solution at $(0, 0)$?

ii. (3 pts) For what values, if any, of k will the equilibrium solution be a saddle?

iii. (3 pts) For what values, if any, of k will the equilibrium solution be a degenerate or star node?

iv. (3 pts) For what values, if any, of k will the trajectories in the phase plane be closed loops?

SOLUTION:

(a)

$$\begin{vmatrix} -1-\lambda & 1 \\ -9 & 5-\lambda \end{vmatrix} = (-1-\lambda)(5-\lambda)+9 = \lambda^2-4\lambda+4 = (\lambda-2)^2 = 0 \implies \lambda = 2 \text{ with multiplicity } 2$$

We need to find nontrivial solutions to $(\mathbf{A}-2\mathbf{I})\vec{\mathbf{v}} = \vec{\mathbf{0}}$ giving

$$\left[\begin{array}{cc|c} -3 & 1 & 0 \\ -9 & 3 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 \end{array}\right] \implies v_1 = \frac{1}{3}v_2 \implies \vec{\mathbf{v}} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Since there is only one eigenvector, we need to find the generalized eigenvector by finding a nontrivial solution to $(\mathbf{A}-2\mathbf{I})\vec{\mathbf{u}} = \vec{\mathbf{v}}$.

$$\left[\begin{array}{cc|c} -3 & 1 & 1 \\ -9 & 3 & 3 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 \end{array}\right] \implies u_1 = -\frac{1}{3}+\frac{1}{3}u_2 \implies \vec{\mathbf{u}} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

The general solution is

$$\vec{\mathbf{x}}(t) = c_1 e^{2t}\begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 e^{2t}\left(t\begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 4 \end{bmatrix}\right)$$

Applying the initial condition yields

$$c_1\begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

with Cramer's Rule giving

$$c_1 = \frac{\begin{vmatrix} 2 & 1 \\ -1 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix}} = \frac{9}{1} = 9 \qquad c_2 = \frac{\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix}} = \frac{-7}{1} = -7$$

and the final solution to the initial value problem as

$$\vec{\mathbf{x}} = e^{2t}\begin{bmatrix} 2-7t \\ -1-21t \end{bmatrix}$$

(b) We have $\text{Tr}\,\mathbf{A} = 2k$, $|\mathbf{A}| = k^2+2$ and $(\text{Tr}\,\mathbf{A})^2 - 4|\mathbf{A}| = 4k^2 - 4(k^2+2) = -8$

i. **[All real numbers]** Since $|\mathbf{A}| = k^2+2 \neq 0$ for all k, the system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{0}}$ has only the trivial solution for all values of k. Thus, regardless of the value of k, the system will always have a unique equilibrium solution at $(0,0)$.

ii. **[None]** $|\mathbf{A}| = k^2+2 > 0$ for all k.

iii. **[None]** $(\text{Tr}\,\mathbf{A})^2 - 4|\mathbf{A}|$ is never 0.

iv. **[0]** $\text{Tr}\,\mathbf{A}$ must be zero and $|\mathbf{A}|$ must be positive.

■