1. [2360/07222 (30 pts)] Solve the initial value problem $\ddot{x} + 2\dot{x} + 10x = \delta(t - 3), \ x(0) = 0, \ \dot{x}(0) = 0.$

**SOLUTION:**

\[ s^2X(s) - sx(0) - x'(0) + 2[sX(s) - x(0)] + 10X(s) = e^{-3s} \]

\[(s^2 + 2s + 10)X(s) = e^{-3s} \] (complete the square)

\[ X(s) = \frac{e^{-3s}}{(s + 1)^2 + 9} = \frac{1}{3} \left(\frac{3}{(s + 1)^2 + 9}\right) e^{-3s} \]

\[ x(t) = L^{-1} \left\{ \frac{1}{3} \left(\frac{3}{(s + 1)^2 + 9}\right) e^{-3s} \right\} = \frac{1}{3} e^{-(t-3)} \sin[3(t - 3)] \text{step}(t - 3) \]

2. [2360/07222 (20 pts)] Write the word **TRUE** or **FALSE** as appropriate. No work need be shown, no work will be graded and no partial credit will be given. Please arrange your answers in a 10 \times 2 matrix with the letters (a)-(j) in column 1 and your answers to each part in the second column.

(a) The parabola $y = x^2$ is a subspace of $\mathbb{R}^2$.

(b) If $B$ is an $m \times n$ matrix, then $|BB^T|$ is defined.

(c) The matrix $A = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is in RREF and therefore the system from which this augmented matrix was derived has a unique solution.

(d) The integrating factor for the equation $(t^2 + 1)y' + 2ty = \cos t$ is $e^{t^2}$.

(e) The isoclines of $y' + y^2 = t$ are parabolas opening downward.

(f) The following system has no equilibrium points:

\[
\begin{align*}
x' &= x^2 + y^2 + 1 \\
y' &= y^4 + \sqrt{x + 1}
\end{align*}
\]

(g) The equation $y' = y^2 - y$ has a stable equilibrium solution at $y = 1$.

(h) There is only one value of $b$ for which the harmonic oscillator governed by the differential equation $4\ddot{x} + b^2\dot{x} + 36x = \cos 3t$ will have unbounded solutions.

(i) Consider the initial value problem $y' = f(t, y), \ y(1) = 1$ with $f(1, 1) = 2$ and $f_y(1, 1)$ not defined. Picard’s theorem guarantees that the IVP does not have a unique solution.

(j) Newton’s Law of Cooling for a certain situation is given by $\frac{dT}{dt} = 2(50 - T)$. With the change of variable $y = 50 - T$, this is equivalent to an exponential decay problem.

**SOLUTION:**

(a) **FALSE** The only subspaces of $\mathbb{R}^2$ are lines through the origin. More demonstrably, $(1, 1)$ and $(2, 4)$ are on the parabola but $(1, 1) + (2, 4) = (3, 5)$ is not, showing that the set (the parabola) is not closed under vector addition.

(b) **TRUE** $B$ is $m \times n$ implying that $B^T$ is $n \times m$ further implying that $BB^T$ is square ($m \times m$) and thus $|BB^T|$ is defined.

(c) **FALSE** The matrix is in RREF but there is a nonpivot column so a solution exists but is not unique.

(d) **FALSE** The integrating factor is $t^2 + 1$.

(e) **FALSE** The isoclines are $t - y^2 = k$, parabolas that open to the right (axis of symmetry along the $x$-axis).

(f) **TRUE** The system has no nullclines and thus no equilibrium points.

(g) **FALSE** The equilibrium solution at $y = 1$ is unstable.

(h) **TRUE** The oscillator must be undamped, so $b = 0$ only.

(i) **FALSE** Since $f_y(1, 1)$ is undefined, it cannot be continuous in a rectangle containing $(1, 1)$. Thus no conclusions can be drawn from Picard’s theorem.
(j) **TRUE.** The change of variable \( y = 50 - T \) yields \( y' = -2y. \)

3. (35 pts) The following parts are not related.

(a) (10 pts) Consider the initial value problem (IVP) \( ty' - 3(\ln t)^2 e^{-y} = 0, \ y(1) = \ln 8. \)
   i. (8 pts) Find the implicit solution to the IVP.
   ii. (2 pts) Find the explicit solution to the IVP and state the interval over which the solution is valid.

(b) (25 pts) A particular solution to \( L(\bar{y}) = f, \) where \( L \) is a linear operator, is \( y_p = \cos t. \) Suppose the characteristic equation for the associated homogeneous equation is \((r - 2)(r^2 - 1) = 0.\) Use Cramer’s Rule to find the solution to the following initial value problem. No points for using other methods.

\[
L(\bar{y}) = f, \quad y(0) = 4, \quad y'(0) = 0, \quad y''(0) = -1
\]

**SOLUTION:**

(a) i. The equation is separable.

\[
\int e^y \, dy = \int \frac{3(\ln t)^2}{t} \, dt \quad (u = \ln t)
\]

\[
e^y = 3 \int u^2 \, du = (\ln t)^3 + C
\]

\[
e^{\ln 8} = (\ln 1)^3 + C \implies C = 8
\]

\[
e^y = (\ln t)^3 + 8
\]

ii. The explicit solution is \( y = \ln \left[(\ln t)^3 + 8\right]. \) Clearly, \( t > 0 \) for input into the “inner” \( \ln t. \) For input into the “outer” natural logarithm function, we also need

\[
(\ln t)^3 + 8 > 0 \implies \ln t > \sqrt[3]{-8} = -2 \implies t > e^{-2}
\]

The solution is valid on \( (e^{-2}, \infty) \).

(b) Based on the characteristic equation, \((r - 2)(r + 1)(r - 1) = 0,\) the solution to the homogeneous equation is \( y_h = c_1 e^{2t} + c_2 e^{-t} + c_3 e^t \) so the general solution to which we apply the initial conditions is \( y = y_h + y_p. \)

\[
y(t) = c_1 e^{2t} + c_2 e^{-t} + c_3 e^t + \cos t
\]

\[
y'(t) = 2c_1 e^{2t} - c_2 e^{-t} + c_3 e^t - \sin t
\]

\[
y''(t) = 4c_1 e^{2t} + c_2 e^{-t} + c_3 e^t - \cos t
\]

At \( t = 0 \) we have

\[
c_1 + c_2 + c_3 + 1 = 4
\]

\[
2c_1 - c_2 + c_3 = 0 \implies \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 4 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}
\]

\[
4c_1 + c_2 + c_3 - 1 = -1
\]

Now use Cramer’s Rule

\[
\begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 4 & 1 & 1 \end{vmatrix} = 1(-1)^{1+1} \begin{vmatrix} -1 & 1 \\ 4 & 1 \end{vmatrix} + 1(-1)^{1+2} \begin{vmatrix} 2 & 1 \\ 4 & 1 \end{vmatrix} + 1(-1)^{1+3} \begin{vmatrix} 2 & -1 \\ 4 & 1 \end{vmatrix} = -2 + (-1)(-2) + 6 = 6
\]

\[
c_1 = \frac{3 \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} - 1 \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} + 1 \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}}{6} = \frac{3(-1)^{1+1} - 1 \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}}{6} = \frac{3(-2)}{6} = -1
\]

\[
c_2 = \frac{3 \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} - 1 \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} + 1 \begin{vmatrix} 1 & 3 \\ 4 & 1 \end{vmatrix}}{6} = \frac{3(-1)^{1+1} - 1 \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}}{6} = \frac{3(-2)}{6} = 1
\]

\[
c_3 = \frac{3 \begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 4 & 0 \end{vmatrix} + 1 \begin{vmatrix} 1 & 1 \\ 4 & 1 \end{vmatrix}}{6} = \frac{3(-1)^{1+1} - 1 \begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix}}{6} = \frac{3(6)}{6} = 3
\]
The solution to the initial value problem is thus

\[ y(t) = -e^{2t} + e^{-t} + 3e^t + \cos t \]

4. [2360/072222 (29 pts)] The following parts are not related.

(a) (12 pts) Consider the function \( f(t) = \begin{cases} 
0 & t < 0 \\
2t & 0 \leq t < 2 \\
5 - t & 2 \leq t 
\end{cases} \)

i. (3 pts) Graph the function.

ii. (4 pts) Write the \( f(t) \) as a single function using step functions.

iii. (5 pts) Find the Laplace transform of \( f(t) \).

(b) (17 pts) Three 200 liter (L) tanks, 1, 2, 3, contain solution that is always well-mixed. Initially, tanks 1 and 3 are half full of pure water and tank 2 is filled with water containing 10 grams (g) of dissolved sugar. Water with 2 g per liter (g/L) of dissolved sugar enters tank 1 at a rate of 1 liter per hour (L/h) and flows through the system as shown in the figure below.

i. (15 pts) Set up, but do NOT solve, a system of differential equations that models the aforementioned scenario. Write your final answer using matrices and vectors.

ii. (2 pts) Over what interval of \( t \) will the solution be valid? Hint: You do not need to solve the system to answer this.

[SOLUTION:]

(a) i. Sketch.

\[ f(t) = t^2 \text{ step}(t) - t^2 \text{ step}(t - 2) + (5 - t) \text{ step}(t - 2) \]

\[ = t^2 \left[ \text{ step}(t) - \text{ step}(t - 2) \right] + (5 - t) \text{ step}(t - 2) \]

\[ = t^2 \text{ step}(t) + (5 - t - t^2) \text{ step}(t - 2) \]

iii. Proceeding term-by-term:

\[ \mathcal{L} \left\{ t^2 \text{ step}(t) \right\} = e^{-0s} \mathcal{L} \left\{ t^2 \right\} = \frac{2}{s^3} \]

\[ \mathcal{L} \left\{ t^2 \text{ step}(t - 2) \right\} = e^{-2s} \mathcal{L} \left\{ (t + 2)^2 \right\} = e^{-2s} \mathcal{L} \left\{ t^2 + 4t + 4 \right\} = e^{-2s} \left( \frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s} \right) \]

\[ \mathcal{L} \left\{ (5 - t) \text{ step}(t - 2) \right\} = \mathcal{L} \left\{ [3 - (t - 2)] \text{ step}(t - 2) \right\} \]

\[ = \mathcal{L} \left\{ 3 \text{ step}(t - 2) \right\} - \mathcal{L} \left\{ (t - 2) \text{ step}(t - 2) \right\} \]

\[ = \frac{3e^{-2s}}{s} - \frac{e^{-2s}}{s^2} = e^{-2s} \left( \frac{3}{s} - \frac{1}{s^2} \right) \]
Thus
\[
\mathcal{L} \left\{ t^2 \text{step}(t) - t^2 \text{step}(t-2) + (5-t) \text{step}(t-2) \right\} = \frac{2}{s^3} e^{-2s} \left( \frac{2}{s^2} + \frac{4}{s} \right) + e^{-2s} \left( \frac{3}{s} - \frac{1}{s^2} \right) = \frac{2}{s^3} e^{-2s} \left( \frac{2}{s^2} + \frac{5}{s^2} + \frac{5}{s} \right)
\]

(b) i. Let \((b)\) represent the mass (grams) of sugar and \(V_1(t), V_2(t), V_3(t)\) the volume of solution (L) in Tank 1, 2, 3 at time \(t\), respectively. Then with
\[
\frac{dV}{dt} = \text{flow rate in} - \text{flow rate out}
\]
we have
\[
\begin{align*}
\frac{dV_1}{dt} &= 1 + 4 - 6 = -1 \quad V_1(0) = 100 \implies \int dV_1 = \int -1 \, dt \implies V_1(t) = 100 - t \\
\frac{dV_2}{dt} &= 6 - 6 = 0 \quad V_2(0) = 200 \implies \int dV_2 = \int 0 \, dt \implies V_2(t) = 200 \\
\frac{dV_3}{dt} &= 6 - 1 - 4 = 1 \quad V_3(0) = 100 \implies \int dV_3 = \int 1 \, dt \implies V_3(t) = 100 + t
\end{align*}
\]

For the amount of sugar in each tank, we will use
\[
\frac{dx}{dt} = \text{(flow rate in)}(\text{concentration in}) - \text{(flow rate out})(\text{concentration out})
\]
\[
\begin{align*}
\frac{dx_1}{dt} &= (1)(2) + 4 \left( \frac{x_3}{100 + t} \right) - 6 \left( \frac{x_1}{100 - t} \right) \\
\frac{dx_2}{dt} &= 6 \left( \frac{x_1}{100 - t} \right) - 6 \left( \frac{x_2}{200} \right) \\
\frac{dx_3}{dt} &= 6 \left( \frac{x_2}{200} \right) - 4 \left( \frac{x_3}{100 + t} \right) - 1 \left( \frac{x_3}{100 + t} \right)
\end{align*}
\]
\[
\begin{bmatrix}
x_1' \\
x_2' \\
x_3'
\end{bmatrix} =
\begin{bmatrix}
-6 \\
6 \\
0
\end{bmatrix}
\begin{bmatrix}
\frac{100 - t}{100} \\
\frac{100 - t}{100} \\
\frac{3}{100}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
+ \begin{bmatrix} 4 \\ 0 \\ -5 \\ \end{bmatrix}
\begin{bmatrix}
\frac{100 + t}{100 + t} \\
\frac{100 + t}{100 + t} \\
\frac{100 + t}{100 + t}
\end{bmatrix}
\begin{bmatrix}
x_1(0) \\
x_2(0) \\
x_3(0)
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

ii. Tank 1 empties and Tank 3 fills after 100 hours so the interval over which the solution to the system is valid is \([0, 100]\).

5. [2360/072222 (36 pts)] The following parts are not related.

(a) (24 pts) Solve the initial value problem \(\vec{x}' = \begin{bmatrix} -1 & 1 \\ -9 & 5 \end{bmatrix} \vec{x}, \quad \vec{x}(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}\). Write your final answer as a single vector.

(b) (12 pts) Consider the linear system \(\vec{x}' = A \vec{x}\) where \(A = \begin{bmatrix} k & 1 \\ -2 & k \end{bmatrix}\) and \(k\) is a real number.

i. (3 pts) For what values, if any, of \(k\) does the system have a unique equilibrium solution at \((0, 0)\)?

ii. (3 pts) For what values, if any, of \(k\) will the equilibrium solution be a saddle?

iii. (3 pts) For what values, if any, of \(k\) will the equilibrium solution be a degenerate or star node?

iv. (3 pts) For what values, if any, of \(k\) will the trajectories in the phase plane be closed loops?

**SOLUTION:**
(a)  
\[
\begin{vmatrix}
-1 - \lambda & 1 \\
-9 & 5 - \lambda
\end{vmatrix} = (-1 - \lambda)(5 - \lambda) + 9 = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 = 0 \implies \lambda = 2 \text{ with multiplicity 2}
\]

We need to find nontrivial solutions to \((A - 2I)\vec{y} = \vec{0}\) giving

\[
\begin{bmatrix}
-3 & 1 & 0 \\
-9 & 3 & 0
\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}
1 & -\frac{1}{3} & 0 \\
0 & 0 & 0
\end{bmatrix} \implies v_1 = \frac{1}{3}v_2 \implies \vec{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}
\]

Since there is only one eigenvector, we need to find the generalized eigenvector by finding a nontrivial solution to \((A - 2I)\vec{u} = \vec{v}\).

\[
\begin{bmatrix}
-3 & 1 & 1 \\
-9 & 3 & 3
\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}
1 & -\frac{1}{3} & -\frac{1}{3} \\
0 & 0 & 0
\end{bmatrix} \implies u_1 = -\frac{1}{3} + \frac{1}{3}u_2 \implies \vec{u} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}
\]

The general solution is

\[
\vec{x}(t) = c_1 e^{2t} \begin{bmatrix}
1 \\ 3
\end{bmatrix} + c_2 e^{2t} \left( t \begin{bmatrix}
1 \\ 3
\end{bmatrix} + \begin{bmatrix} 1 \\ 4 \end{bmatrix} \right)
\]

Applying the initial condition yields

\[
c_1 \begin{bmatrix}
1 \\ 3
\end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix}
2 \\ -1
\end{bmatrix}
\]

with Cramer’s Rule giving

\[
c_1 = \frac{2 - 1}{-1 - 4} = \frac{9}{5} = 9, \quad c_2 = \frac{1 - 2}{1 - 4} = -\frac{7}{3} = -7
\]

and the final solution to the initial value problem as

\[
\vec{x} = e^{2t} \begin{bmatrix}
2 - 7t \\ -1 - 21t
\end{bmatrix}
\]

(b) We have \(\text{Tr} A = 2k, |A| = k^2 + 2\) and \((\text{Tr} A)^2 - 4|A| = 4k^2 - 4(k^2 + 2) = -8\)

i. **All real numbers** Since \(|A| = k^2 + 2 \neq 0\) for all \(k\), the system \(A\vec{x} = \vec{0}\) has only the trivial solution for all values of \(k\). Thus, regardless of the value of \(k\), the system will always have a unique equilibrium solution at \((0, 0)\).

ii. **None** \(|A| = k^2 + 2 > 0\) for all \(k\).

iii. **None** \((\text{Tr} A)^2 - 4|A|\) is never 0.

iv. **0** \(\text{Tr} A\) must be zero and \(|A|\) must be positive.