- This exam is worth 150 points and has 5 questions.
- **Show all work and simplify your answers!** Answers with no justification will receive no points unless otherwise noted.
- **Please begin each problem on a new page.**
- **DO NOT** leave the exam until you have satisfactorily scanned and uploaded your exam to Gradescope.
- You are taking this exam in a proctored and honor code enforced environment. **NO** calculators, cell phones, or other electronic devices or the internet are permitted. You are allowed one 8.5”× 11” crib sheet with writing on both sides.

At the top of the page containing your answer to problem 1, write the following italicized statement and sign your name to it: “*I will abide by the CU Boulder Honor Code on this exam.*” **FAILURE TO INCLUDE THIS MAY RESULT IN A PENALTY.**

1. [2360/072222 (30 pts)] Solve the initial value problem $\ddot{x} + 2\dot{x} + 10x = \delta(t-3),\ x(0) = 0,\ \dot{x}(0) = 0.$

2. [2360/072222 (20 pts)] Write the <u>word</u> **TRUE** or **FALSE** as appropriate. No work need be shown, no work will be graded and no partial credit will be given. Please arrange your answers in a 10×2 matrix with the letters (a)-(j) in column 1 and your answers to each part in the second column.

   (a) The parabola $y = x^2$ is a subspace of $\mathbb{R}^2$.

   (b) If $\mathbf{B}$ is an $m \times n$ matrix, then $\left|\mathbf{B}\mathbf{B}^{\mathrm{T}}\right|$ is defined.

   (c) The matrix $\mathbf{A} = \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ is in RREF and therefore the system from which this augmented matrix was derived has a unique solution.

   (d) The integrating factor for the equation $(t^2+1)y' + 2ty = \cos t$ is e^{t^2}.

   (e) The isoclines of $y' + y^2 = t$ are parabolas opening downward.

   (f) The following system has no equilibrium points:

   $$\begin{aligned} x' &= x^2 + y^2 + 1 \\ y' &= y^4 + \sqrt{x+1} \end{aligned}$$

   (g) The equation $y' = y^2 - y$ has a stable equilibrium solution at $y = 1$.

   (h) There is only one value of b for which the harmonic oscillator governed by the differential equation $4\ddot{x} + b^2\dot{x} + 36x = \cos 3t$ will have unbounded solutions.

   (i) Consider the initial value problem $y' = f(t,y),\ y(1) = 1$ with $f(1,1) = 2$ and $f_y(1,1)$ not defined. Picard’s theorem guarantees that the IVP does not have a unique solution.

   (j) Newton’s Law of Cooling for a certain situation is given by $\dfrac{\mathrm{d}T}{\mathrm{d}t} = 2(50 - T)$. With the change of variable $y = 50 - T$, this is equivalent to an exponential decay problem.

3. (35 pts) The following parts are not related.

   (a) (10 pts) Consider the initial value problem (IVP) $\quad ty' - 3(\ln t)^2 e^{-y} = 0, \quad y(1) = \ln 8.$

      i. (8 pts) Find the implicit solution to the IVP.

      ii. (2 pts) Find the explicit solution to the IVP and state the interval over which the solution is valid.

   (b) (25 pts) A particular solution to $L(\vec{\mathbf{y}}) = f$, where L is a linear operator, is $y_p = \cos t$. Suppose the characteristic equation for the associated homogeneous equation is $(r-2)(r^2-1) = 0$. Use Cramer’s Rule to find the solution to the following initial value problem. No points for using other methods.

   $$L(\vec{\mathbf{y}}) = f, \quad y(0) = 4,\ y'(0) = 0,\ y''(0) = -1$$

4. [2360/072222 (29 pts)] The following parts are not related.

(a) (12 pts) Consider the function $f(t) = \begin{cases} 0 & t < 0 \\ t^2 & 0 \le t < 2 \\ 5 - t & 2 \le t \end{cases}$

i. (3 pts) Graph the function.

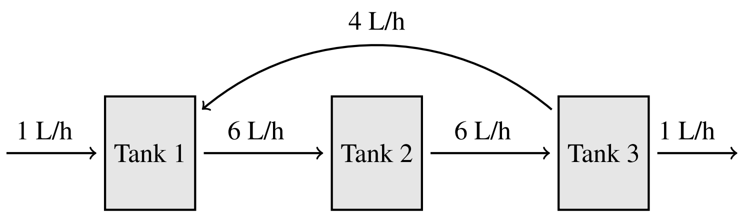
ii. (4 pts) Write the $f(t)$ as a single function using step functions.

iii. (5 pts) Find the Laplace transform of $f(t)$.

(b) (17 pts) Three 200 liter (L) tanks, $1, 2, 3$, contain solution that is always well-mixed. Initially, tanks 1 and 3 are half full of pure water and tank 2 is filled with water containing 10 grams (g) of dissolved sugar. Water with 2 g per liter (g/L) of dissolved sugar enters tank 1 at a rate of 1 liter per hour (L/h) and flows through the system as shown in the figure below.

i. (15 pts) Set up, but do **NOT** solve, a system of differential equations that models the aforementioned scenario. Write your final answer using matrices and vectors.

ii. (2 pts) Over what interval of t will the solution be valid? Hint: You do not need to solve the system to answer this.



5. [2360/072222 (36 pts)] The following parts are not related.

(a) (24 pts) Solve the initial value problem $\vec{\mathbf{x}}' = \begin{bmatrix} -1 & 1 \\ -9 & 5 \end{bmatrix} \vec{\mathbf{x}}, \quad \vec{\mathbf{x}}(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$. Write your final answer as a single vector.

(b) (12 pts) Consider the linear system $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}}$ where $\mathbf{A} = \begin{bmatrix} k & 1 \\ -2 & k \end{bmatrix}$ and k is a real number.

i. (3 pts) For what values, if any, of k does the system have a unique equilibrium solution at $(0, 0)$?

ii. (3 pts) For what values, if any, of k will the equilibrium solution be a saddle?

iii. (3 pts) For what values, if any, of k will the equilibrium solution be a degenerate or star node?

iv. (3 pts) For what values, if any, of k will the trajectories in the phase plane be closed loops?

Short table of Laplace Transforms: $\mathscr{L}\{f(t)\} = F(s) \equiv \int_0^\infty e^{-st} f(t)\,\mathrm{d}t$

In this table, a, b, c are real numbers with $c \ge 0$, and $n = 0, 1, 2, 3, \ldots$

$$\mathscr{L}\left\{t^n e^{at}\right\} = \frac{n!}{(s-a)^{n+1}} \qquad \mathscr{L}\left\{e^{at}\cos bt\right\} = \frac{s-a}{(s-a)^2+b^2} \qquad \mathscr{L}\left\{e^{at}\sin bt\right\} = \frac{b}{(s-a)^2+b^2}$$

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{\mathrm{d}^n F(s)}{\mathrm{d}s^n} \qquad \mathscr{L}\left\{e^{at} f(t)\right\} = F(s-a) \qquad \mathscr{L}\{\delta(t-c)\} = e^{-cs}$$

$$\mathscr{L}\left\{t f'(t)\right\} = -F(s) - s\frac{\mathrm{d}F(s)}{\mathrm{d}s} \qquad \mathscr{L}\{f(t-c)\,\mathrm{step}(t-c)\} = e^{-cs}F(s) \qquad \mathscr{L}\{f(t)\,\mathrm{step}(t-c)\} = e^{-cs}\mathscr{L}\{f(t+c)\}$$

$$\mathscr{L}\left\{f^{(n)}(t)\right\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - \cdots - f^{(n-1)}(0)$$