1. [2360/070822 (18 pts)] A mass of 1 kg is attached to a spring with restoring/spring constant $k = 25$ newton/m.

   (a) (2 pts) Assuming there is no damping and that the system is driven by the forcing term $20\cos(\omega_f t)$ newtons, find a value of ω_f, if any, that guarantees that the amplitude of the resulting oscillations grows without bound.

   (b) (16 pts) Now assume that the system is driven by the forcing term $102\cos t$ newtons and the damping force is 6 times the instantaneous velocity.

      i. (4 pts) Write down the differential equation for this mass-spring motion.

      ii. (4 pts) Verify that $x_p(t) = 4\cos t + \sin t$ is a particular solution to the differential equation in part (i).

      iii. (6 pts) Find the general solution of the differential equation.

      iv. (2 pts) Is the oscillator underdamped, overdamped or critically damped?

SOLUTION:

(a) $m = 1$, $k = 25$, $b = 0$, $\omega_0 = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{25}{1}} = 5$. For solutions to grow without bound, the oscillator needs to be in pure resonance, that is, $\omega_f = \omega_0 = 5$.

(b) i. If $x(t)$ is the displacement of the mass from its equilibrium position, we have $\ddot{x} + 6\dot{x} + 25x = 102\cos t$.

   ii. $\ddot{x}_p + 6\dot{x}_p + 25x_p = -4\cos t - \sin t + 6(-4\sin t + \cos t) + 25(4\cos t + \sin t) = 102\cos t$

   iii. The characteristic equation is $r^2 + 6r + 25 = 0$ so that

$$r = \frac{-6 \pm \sqrt{6^2 - (4)(1)(25)}}{2} = \frac{-6 \pm \sqrt{-64}}{2} = -3 \pm 4i$$

   The general solution is thus $x(t) = x_h + x_p = e^{-3t}\left(c_1\cos 4t + c_2\sin 4t\right) + 4\cos t + \sin t$

   iv. $b^2 - 4mk = 6^2 - 4(1)(25) = -64 < 0$ so the oscillator is underdamped.

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2. [2360/070822 (26 pts)] Consider the differential equation $y''' - 4y'' + 5y' = f(t)$.

   (a) (6 pts) Find the solution when $f(t) = 0$.

   (b) (4 pts) For the following $f(t)$, write down the appropriate form of the particular solution that would be used for the Method of Undetermined Coefficients. Do **not** solve for the coefficients. No justification needed and no partial credit available.

      i. $f(t) = t^2e^{3t}$

      ii. $f(t) = 3t\sin t$

      iii. $f(t) = 5e^{-t}\sin t$

      iv. $f(t) = \dfrac{\left(t^2-1\right)(t+2)\sin t}{t+1}$ for $t > 0$

   (c) (16 pts) Now let $f(t) = 40t - 7 - 6e^{2t}$.

      i. (10 pts) Find the particular solution.

      ii. (4 pts) Write the general solution to the differential equation.

      iii. (2 pts) Identify the transient and steady state solutions.

SOLUTION:

(a) The characteristic equation is $r^3 - 4r^2 + 5r = r(r^2 - 4r + 5) = 0 \implies r = 0, 2+i, 2-i$. Thus $y_h = c_1 + e^{2t}(c_2\cos t + c_3\sin t)$.

(b) i. $y_p = (At^2 + Bt + C)e^{3t}$

   ii. $y_p = (At + B)\cos t + (Ct + D)\sin t$

   iii. $y_p = e^{-t}(A\cos t + B\sin t)$

   iv. Note that $f(t)$ can be simplified to $(t-1)(t+2)\sin t = (t^2 + t - 2)\sin t$. Thus

$$y_p = (At^2 + Bt + C)\cos t + (Et^2 + Ft + G)\sin t$$

(c) i. The initial guess for $y_p = A + Bt + Ce^{2t}$. However, since constants are solutions of the differential equation, the first two terms must be multiplied by t giving $y_p = At + Bt^2 + Ce^{2t}$. Then

$$y_p' = A + 2Bt + 2Ce^{2t}$$
$$y_p'' = 2B + 4Ce^{2t}$$
$$y_p''' = 8Ce^{2t}$$

and

$$y_p''' - 4y_p'' + 5y' = 8Ce^{2t} - 4\left(2B + 4Ce^{2t}\right) + 5\left(A + 2Bt + 2Ce^{2t}\right) = 40t - 7 - 6e^{2t}$$
$$(8C - 16C + 10C)e^{2t} + (10B)t + (-8B + 5A) = 40t - 7 - 6e^{2t}$$
$$2Ce^{2t} + 10Bt + (-8B + 5A) = 40t - 7 - 6e^{2t}$$
$$2C = -6 \implies C = -3$$
$$10B = 40 \implies B = 4$$
$$-8(4) + 5A = -7 \implies A = 5$$

Thus $y_p = -3e^{2t} + 4t^2 + 5t$.

ii. $y(t) = y_h(t) + y_p(t) = c_1 + e^{2t}(c_2 \cos t + c_3 \sin t) - 3e^{2t} + 4t^2 + 5t$

iii. There is no transient solution (no part of the solution tends to zero as $t \to \infty$.) There is no steady state solution as the entire solution grows without bound (c_1 will be overwhelmed by the other terms as $t \to \infty$).

3. [2360/070822 (30 pts)] Consider the Euler-Cauchy equation $x^2y'' - 12y = \dfrac{49}{x^3}$, $x > 0$ with initial conditions $y(1) = 5$, $y'(1) = -1$.

(a) (8 pts) Assuming solutions of the form $y(x) = x^r$, solve the associated homogeneous equation.

(b) (6 pts) Show that your solution(s) to part (a) form(s) a basis for the solution space of the homogeneous equation.

(c) (10 pts) Find the general solution to the original nonhomogeneous equation.

(d) (6 pts) Solve the initial value problem.

SOLUTION:

(a) $y = x^r \implies y' = rx^{r-1} \implies y'' = r(r-1)x^{r-2}$. Substituting these into the homogeneous equation gives

$$x^2y'' - 12y = x^2(r)(r-1)x^{r-2} - 12x^r = x^r(r^2 - r - 12) = 0$$

yielding the characteristic equation $r^2 - r - 12 = (r-4)(r+3) = 0 \implies r = -3, 4$. The solutions to the homogeneous equation are thus $y = x^{-3}$ and $y = x^4$ giving the general solution as $y = c_1x^4 + c_2x^{-3}$.

(b) A basis for the solution space will consist of two linearly independent solutions.

$$x^2\left(x^{-3}\right)'' - 12x^{-3} = x^2(-3)(-4)x^{-5} - 12x^{-3} = 0 \quad \checkmark$$
$$x^2\left(x^4\right)'' - 12x^4 = x^2(4)(3)x^2 - 12x^4 = 0 \quad \checkmark$$

So both $y_1 = x^{-3}$ and $y_2 = x^4$ are solutions to the homogeneous equation. Now check for linear independence.

$$W\left[x^{-3}, x^4\right](x) = \begin{vmatrix} x^{-3} & x^4 \\ -3x^{-4} & 4x^3 \end{vmatrix} = 7 \neq 0$$

Thus the two solutions are linearly independent. The dimension of the solution space is 2 since the equation is second order. Therefore, $\left\{x^{-3}, x^4\right\}$ forms a basis for the solution space of the homogeneous equation.

(c) We must use variation of parameters to find the particular solution in the form $y_p = v_1(x)y_1(x) + v_2(x)y_2(x)$. We let $y_1 = x^{-3}$ and $y_2 = x^4$. To use the variation of parameters formula, we need to rewrite the differential equation as $y'' - 12x^{-2}y = 49x^{-5}$ giving $f(x) = 49x^{-5}$. Using the Wronskian from part (b) we have

$$v_1 = \int \frac{-y_2 f}{W[y_1, y_2]}\,\mathrm{d}x = \int \frac{-x^4\left(49x^{-5}\right)}{7}\,\mathrm{d}x = -7\int \frac{\mathrm{d}x}{x} = -7\ln|x| = -7\ln x \quad (x > 0)$$
$$v_2 = \int \frac{y_1 f}{W[y_1, y_2]}\,\mathrm{d}x = \int \frac{x^{-3}\left(49x^{-5}\right)}{7}\,\mathrm{d}x = 7\int x^{-8}\,\mathrm{d}x = -x^{-7}$$
$$y_p = -7x^{-3}\ln x - x^{-7}x^4 = -7x^{-3}\ln x - x^{-3}$$

The general solution to the nonhomogeneous equation is thus

$$y(x) = c_1 x^{-3} + c_2 x^4 - 7x^{-3} \ln x$$

where the x^{-3} term in the particular solution has been absorbed into the c_1 term.

(d) Apply the initial conditions. We need $y'(x) = -3c_1 x^{-4} + 4c_2 x^3 - 7x^{-4} + 21x^{-4} \ln x$.

$$y(1) = c_1 + c_2 = 5$$
$$y'(1) = -3c_1 + 4c_2 - 7 = -1$$

Using Cramer's Rule

$$c_1 = \frac{\begin{vmatrix} 5 & 1 \\ -1 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ -3 & 4 \end{vmatrix}} = \frac{21}{7} = 3 \qquad c_2 = \frac{\begin{vmatrix} 1 & 5 \\ -3 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ -3 & 4 \end{vmatrix}} = \frac{14}{7} = 2$$

Thus the solution to the initial value problem is

$$y(x) = 2x^{-3} + 3x^4 - 7x^{-3} \ln x = 3x^4 + x^{-3} (2 - 7 \ln x)$$

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4. [2360/070822 (12 pts)] The following parts are not related.

(a) (8 pts) Convert the following initial value problem into a system of first order initial value problems. If possible, write your answer using matrices.

$$\cos t \frac{\mathrm{d}^4 y}{\mathrm{d}t^4} - 2e^t \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = 3 \quad y(1) = 1,\ y'(1) = -1,\ y''(1) = 2,\ y'''(1) = 0 \quad 0 \le t < \pi/2$$

(b) (4 pts) Which of the following differential equations describes a conservative system? No explanation necessary and no partial credit available.

i. $\ddot{x} = (\dot{x})^2$

ii. $\ddot{x} = x^2 + t^2$

iii. $3\ddot{x} = -5x^2$

iv. $2\ddot{x} + 2\dot{x} + 3x = 0$

SOLUTION:

(a) Let $u_1 = y$, $u_2 = y'$, $u_3 = y''$, and $u_4 = y'''$. Then

$$u_1' = y' = u_2$$
$$u_2' = y'' = u_3$$
$$u_3' = y''' = u_4$$
$$u_4' = y^{(4)} = \left(3 + 2e^t y''\right) / \cos t = \left(2e^t \sec t\right) u_3 + 3 \sec t$$

Since this a linear equation, we can write it using matrices as

$$\begin{bmatrix} u_1' \\ u_2' \\ u_3' \\ u_4' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 2e^t \sec t & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3 \sec t \end{bmatrix} \qquad \begin{bmatrix} u_1(1) \\ u_2(1) \\ u_3(1) \\ u_4(1) \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix}$$

(b) Conservative systems must be governed by equations of the form $m\ddot{x} + V'(x) = 0$. Thus iii is the only choice; i and iv are damped and ii is non-autonomous.

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5. [2360/070822 (14 pts) The following parts are not related.

(a) (6 pts) Consider the linear operator

$$L(\vec{\mathbf{y}}) = a_9(t)y^{(9)}(t) + a_8(t)y^{(8)}(t) + \cdots + a_1(t)y'(t) + a_0(t)y(t)$$

where $a_0(t), a_1(t), \ldots, a_9(t)$ are continuous on some interval I.

i. (3 pts) Is the set of solutions to $L(\vec{\mathbf{y}}) = 1$ a vector space? Justify your answer.

ii. (3 pts) Let $y_1(t), y_2(t), \ldots, y_{10}(t)$ be solutions to $L(\vec{\mathbf{y}}) = 0$. Is the set $\{y_1, y_2, \ldots, y_{10}\}$ a basis for the solution space of $L(\vec{\mathbf{y}}) = 0$? Justify your answer.

(b) (8 pts) Find the general solution of $y^{(5)} + 2y^{(4)} + y''' = 0$.

SOLUTION:

(a) i. No. $\vec{\mathbf{0}}$ is not in the set since the equation is nonhomogeneous.

ii. The set consists of 10 vectors and the dimension of the solution space is 9 since we are dealing with a ninth order differential equation. The vectors are thus linearly dependent and cannot form a basis for the space.

(b) The characteristic equation is

$$r^5 + 2r^4 + r^3 = r^3(r^2 + 2r + 1) = r^3(r+1)^2 = 0$$

which has roots $r = 0$ with multiplicity 3 and $r = -1$ with multiplicity 2. The general solution is

$$y(t) = c_1 + c_2 t + c_3 t^2 + c_4 e^{-t} + c_5 t e^{-t}$$

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