- This exam is worth 100 points and has 5 questions.
- **Show all work and simplify your answers!** Answers with no justification will receive no points unless otherwise noted.
- **Please begin each problem on a new page.**
- **DO NOT** leave the exam until you have satisfactorily scanned and uploaded your exam to Gradescope.
- You are taking this exam in a proctored and honor code enforced environment. **NO** calculators, cell phones, or other electronic devices or the internet are permitted. You are allowed one 8.5"× 11" crib sheet with writing on one side.

At the top of the page containing your answer to problem 1, write the following italicized statement and sign your name to it:
"*I will abide by the CU Boulder Honor Code on this exam.*" **FAILURE TO INCLUDE THIS MAY RESULT IN A PENALTY.**

1. [2360/070822 (18 pts)] A mass of 1 kg is attached to a spring with restoring/spring constant $k = 25$ newton/m.

   (a) (2 pts) Assuming there is no damping and that the system is driven by the forcing term $20\cos(\omega_f t)$ newtons, find a value of ω_f, if any, that guarantees that the amplitude of the resulting oscillations grows without bound.

   (b) (16 pts) Now assume that the system is driven by the forcing term $102\cos t$ newtons and the damping force is 6 times the instantaneous velocity.

   i. (4 pts) Write down the differential equation for this mass-spring motion.
   ii. (4 pts) Verify that $x_p(t) = 4\cos t + \sin t$ is a particular solution to the differential equation in part (i).
   iii. (6 pts) Find the general solution of the differential equation.
   iv. (2 pts) Is the oscillator underdamped, overdamped or critically damped?

2. [2360/070822 (26 pts)] Consider the differential equation $y''' - 4y'' + 5y' = f(t)$.

   (a) (6 pts) Find the solution when $f(t) = 0$.

   (b) (4 pts) For the following $f(t)$, write down the appropriate form of the particular solution that would be used for the Method of Undetermined Coefficients. Do **not** solve for the coefficients. No justification needed and no partial credit available.

   i. $f(t) = t^2 e^{3t}$
   ii. $f(t) = 3t\sin t$
   iii. $f(t) = 5e^{-t}\sin t$
   iv. $f(t) = \dfrac{\left(t^2-1\right)(t+2)\sin t}{t+1}$ for $t > 0$

   (c) (16 pts) Now let $f(t) = 40t - 7 - 6e^{2t}$.

   i. (10 pts) Find the particular solution.
   ii. (4 pts) Write the general solution to the differential equation.
   iii. (2 pts) Identify the transient and steady state solutions.

3. [2360/070822 (30 pts)] Consider the Euler-Cauchy equation $x^2y'' - 12y = \dfrac{49}{x^3}$, $x > 0$ with initial conditions $y(1) = 5$, $y'(1) = -1$.

   (a) (8 pts) Assuming solutions of the form $y(x) = x^r$, solve the associated homogeneous equation.

   (b) (6 pts) Show that your solution(s) to part (a) form(s) a basis for the solution space of the homogeneous equation.

   (c) (10 pts) Find the general solution to the original nonhomogeneous equation.

   (d) (6 pts) Solve the initial value problem.

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4. [2360/070822 (12 pts)] The following parts are not related.

(a) (8 pts) Convert the following initial value problem into a system of first order initial value problems. If possible, write your answer using matrices.

$$\cos t \frac{\mathrm{d}^4 y}{\mathrm{d}t^4} - 2e^t \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = 3 \quad y(1) = 1,\ y'(1) = -1,\ y''(1) = 2,\ y'''(1) = 0 \quad 0 \le t < \pi/2$$

(b) (4 pts) Which of the following differential equations describes a conservative system? No explanation necessary and no partial credit available.

i. $\ddot{x} = (\dot{x})^2$

ii. $\ddot{x} = x^2 + t^2$

iii. $3\ddot{x} = -5x^2$

iv. $2\ddot{x} + 2\dot{x} + 3x = 0$

5. [2360/070822 (14 pts) The following parts are not related.

(a) (6 pts) Consider the linear operator

$$L(\vec{\mathbf{y}}) = a_9(t)y^{(9)}(t) + a_8(t)y^{(8)}(t) + \cdots + a_1(t)y'(t) + a_0(t)y(t)$$

where $a_0(t), a_1(t), \ldots, a_9(t)$ are continuous on some interval I.

i. (3 pts) Is the set of solutions to $L(\vec{\mathbf{y}}) = 1$ a vector space? Justify your answer.

ii. (3 pts) Let $y_1(t), y_2(t), \ldots, y_{10}(t)$ be solutions to $L(\vec{\mathbf{y}}) = 0$. Is the set $\{y_1, y_2, \ldots, y_{10}\}$ a basis for the solution space of $L(\vec{\mathbf{y}}) = 0$? Justify your answer.

(b) (8 pts) Find the general solution of $y^{(5)} + 2y^{(4)} + y''' = 0$.