1. [2360/062422 (14 pts)] Consider the linear system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ where

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 1 \\ -2 & 1 & 0 \\ 2 & -3 & 4 \end{bmatrix}, \quad \vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \vec{\mathbf{b}} = \begin{bmatrix} 2 \\ -2 \\ 6 \end{bmatrix}$$

(a) (6 pts) Calculate $|\mathbf{A}|$ using the cofactor expansion method.

(b) (2 pts) Is there a unique solution to $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$? Justify your answer.

(c) (6 pts) The RREF of the augmented matrix for the system is $\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$. Find the solution to the original system, writing your answer in the form $\vec{\mathbf{x}} = \vec{\mathbf{x}}_h + \vec{\mathbf{x}}_p$, clearly labeling $\vec{\mathbf{x}}_h$ and $\vec{\mathbf{x}}_p$.

SOLUTION:

(a) Expanding along the third column:

$$\begin{aligned} |\mathbf{A}| &= 1(-1)^{1+3}\begin{vmatrix} -2 & 1 \\ 2 & -3 \end{vmatrix} + 4(-1)^{3+3}\begin{vmatrix} 1 & -1 \\ -2 & 1 \end{vmatrix} \\ &= 1(1)[(-2)(-3) - (2)(1)] + 4(1)[(1)(1) - (-2)(-1)] \\ &= (1)(4) + 4(-1) = 0 \end{aligned}$$

(b) Since $|\mathbf{A}| = 0$, there can be no unique solution.

(c) Using the RREF as given, the nonpivot column corresponds to x_3 so it is a free variable which we set to t. Then $x_2 = -2 + 2t$ and $x_1 = t$. The solution is thus

$$\vec{\mathbf{x}} = \begin{bmatrix} t \\ -2+2t \\ t \end{bmatrix} = \underbrace{t\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}}_{\vec{\mathbf{x}}_h} + \underbrace{\begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}}_{\vec{\mathbf{x}}_p}, \; t \in \mathbb{R}$$

2. [2360/062422 (12 pts)] Solve the following linear system by finding the inverse of an appropriate matrix.

$$\begin{aligned} x_1 + x_3 &= 2 \\ 2x_1 - 4x_2 + 6x_3 &= 4 \\ 3x_1 + 2x_2 - x_3 &= 4 \end{aligned}$$

SOLUTION:

The system of equations can be written in the form $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ as

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & -4 & 6 \\ 3 & 2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & -4 & 6 & 0 & 1 & 0 \\ 3 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3^*=-3R_1+R_3]{R_2^*=-2R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -4 & 4 & -2 & 1 & 0 \\ 0 & 2 & -4 & -3 & 0 & 1 \end{array}\right] \xrightarrow[R_2 \leftrightarrow R_3]{R_2^*=2R_3+R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & -4 & -3 & 0 & 1 \\ 0 & 0 & -4 & -8 & 1 & 2 \end{array}\right] \xrightarrow[R_1^*=4R_1+R3]{R_2^*=-1R_3+R_2} \left[\begin{array}{ccc|ccc} 4 & 0 & 0 & -4 & 1 & 2 \\ 0 & 2 & 0 & 5 & -1 & -1 \\ 0 & 0 & -4 & -8 & 1 & 2 \end{array}\right] \xrightarrow{\substack{R_1^*=\frac{1}{4}R_1 \\ R_2^*=\frac{1}{2}R_2 \\ R_3^*=-\frac{1}{4}R3}}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & \frac{1}{4} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{5}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & 2 & -\frac{1}{4} & -\frac{1}{2} \end{array}\right] \implies \mathbf{A}^{-1} = \begin{bmatrix} -1 & \frac{1}{4} & \frac{1}{2} \\ \frac{5}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 2 & -\frac{1}{4} & -\frac{1}{2} \end{bmatrix}$$

Thus

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 & \frac{1}{4} & \frac{1}{2} \\ \frac{5}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 2 & -\frac{1}{4} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

3. [2360/062422 (30 pts)] The following parts are not related.

(a) (6 pts) Consider the matrix $\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ 2 & 1 & -5 \\ 6 & -4 & k \end{bmatrix}$. Find the value of k, if any, such that $\vec{\mathbf{x}} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$ is an eigenvector of $\mathbf{A}$ associated with the eigenvalue 5. Do not find any other eigenvalues or eigenvectors.

(b) (9 pts) Consider the matrix $\mathbf{A} = \begin{bmatrix} (b-1) & 0 & 0 \\ 7 & (b+1) & 0 \\ -3 & 2 & b^2 \end{bmatrix}$.

i. (3 pts) Find all values of b such that the matrix has 0 as an eigenvalue.

ii. (3 pts) What is $|\mathbf{A}|$ for the values of b found in part (i)?

iii. (3 pts) For the values of b found in part (i), is the system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{0}}$, where $\vec{\mathbf{x}} \in \mathbb{R}^3$, consistent? Explain briefly.

(c) (5 pts) The characteristic polynomial for a certain matrix is $p(\lambda) = \lambda^3 (\lambda+1)^2 (\lambda^2+4)$.

i. (4 pts) Find the eigenvalues of the matrix and state the multiplicity of each.

ii. (1 pt) What is the order/size of the matrix from which the characteristic equation was derived?

(d) (10 pts) Let $\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$.

i. (8 pts) Find a basis for the eigenspace associated with the real eigenvalue of $\mathbf{A}$.

ii. (2 pts) What is the dimension of the eigenspace in part (i)?

SOLUTION:

(a) We need to find k such that $\mathbf{A}\vec{\mathbf{x}} = 5\vec{\mathbf{x}}$. We need

$$\begin{bmatrix} 2 & 1 & 0 \\ 2 & 1 & -5 \\ 6 & -4 & k \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 5 \\ 15 \\ -6-2k \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 5 \\ 15 \\ -10 \end{bmatrix}$$

Thus $-6-2k = -10 \implies k = 2$

(b) i. Since $\mathbf{A}$ is lower triangular, the eigenvalues are simply the diagonal elements, $b-1, b+1, b^2$. Thus, $\mathbf{A}$ will have 0 as an eigenvalue if $b = -1, 0, 1$.

ii. The determinant of a lower triangular matrix is simply the product of the diagonal elements. Thus $|\mathbf{A}| = (b-1)(b+1)b^2$. For the values of b in part (i), this will be 0.

iii. Homogeneous systems are always consistent since they always have at least the trivial solution $\vec{\mathbf{x}} = \vec{\mathbf{0}}$.

(c) i. $\lambda^3 (\lambda+1)^2 (\lambda^2+4) = 0 \implies \lambda = 0, -1, \pm 2i$. 0 has multiplicity 3, -1 has multiplicity 2, and $2i, -2i$ each have multiplicity 1.

ii. The characteristic polynomial has degree 7 so the matrix whose characteristic polynomial is given is order 7 or 7×7.

(d) i.

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} -\lambda & 0 & 1 \\ 0 & -\lambda & 0 \\ -1 & 0 & -\lambda \end{vmatrix} = -\lambda^3 - \lambda = -\lambda(\lambda^2+1) = 0 \implies \lambda = 0, \pm i$$

We seek nontrivial solutions of $(\mathbf{A} - 0\mathbf{I})\,\vec{\mathbf{v}} = \vec{\mathbf{0}}$ where $\vec{\mathbf{v}} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}^{\mathrm{T}}$.

$$\left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{array}\right] \xrightarrow[R_1 \leftrightarrow R_3]{R_3^* = -R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \implies \begin{array}{l} v_1 = 0 \\ v_2 = t \quad t \in \mathbb{R} \\ v_3 = 0 \end{array}$$

A basis for the eigenspace is thus $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

ii. Since the basis consists of a single vector, the dimension of the eigenspace is 1.

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4. [2360/062422 (24 pts)] Let $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$. Evaluate each of the following expressions or explain why it is not defined.

(a) $\mathbf{AB}$ (b) $\mathbf{B} + 2\mathbf{I}$ (c) $\left(\mathbf{A}^{\mathrm{T}}\mathbf{A}\right)^{\mathrm{T}}$ (d) $|\mathbf{A}|\mathbf{A}^{-1}$ (e) $\mathbf{B}^{\mathrm{T}}\mathbf{A}$ (f) $\operatorname{Tr}\left(\mathbf{B}^2\right)$

SOLUTION:

(a)

$$\mathbf{AB} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 0 \\ 1 & -1 \end{bmatrix}$$

(b)

$$\mathbf{B} + 2\mathbf{I} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} + 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}$$

(c) Method 1

$$\left(\mathbf{A}^{\mathrm{T}}\mathbf{A}\right)^{\mathrm{T}} = \left(\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \right)^{\mathrm{T}} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Method 2

$$\left(\mathbf{A}^{\mathrm{T}}\mathbf{A}\right)^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}} \left(\mathbf{A}^{\mathrm{T}}\right)^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}}\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

(d) Not defined. Neither the determinant nor the inverse are defined for nonsquare matrices.

(e) Not defined. $\mathbf{B}^{\mathrm{T}}$ is 2×2 and $\mathbf{A}$ is 3×2 so the matrix multiplication is not possible.

(f)

$$\mathbf{B}^2 = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \implies \operatorname{Tr}\left(\mathbf{B}^2\right) = 0 + (-1) = -1$$

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5. [2360/062422 (20 pts)] The following parts are not related. However, you need to provide justification for all of your answers for each part. Correct answers with missing or incorrect justifications will receive no points.

(a) (14 pts) Consider the vector space $\mathbb{R}^4$.

i. (8 pts) Suppose $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$ are vectors in $\mathbb{R}^4$ and that the only solution to $c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + c_3\vec{\mathbf{v}}_3 = \vec{\mathbf{0}}$ is $c_1 = c_2 = c_3 = 0$.

A. (4 pts) Is $\operatorname{span}\left\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\right\} = \mathbb{R}^4$?

B. (4 pts) Is $\vec{\mathbf{v}}_1 \in \operatorname{span}\left\{\vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\right\}$?

ii. (6 pts) Let $\mathbb{W}$ be the set of vectors $\mathbb{R}^4$ of the form $\begin{bmatrix} a \\ b \\ 0 \\ ab \end{bmatrix}$ where a and b are real numbers. Is $\mathbb{W}$ a subspace of $\mathbb{R}^4$?

(b) (6 pts) Can the set $\left\{(t-2)^2, t^2 - 2, -2\right\}$ be a basis for $\mathbb{P}_2$, the vector space of all polynomials of degree less than or equal to 2?

SOLUTION:

(a) i. Since the only solution to $c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + c_3\vec{\mathbf{v}}_3 = \vec{\mathbf{0}}$ is $c_1 = c_2 = c_3 = 0$ the vectors are linearly independent.

A. The vectors cannot span $\mathbb{R}^4$ since there are only three vectors and the dimension of $\mathbb{R}^4$ is 4.

B. No, since this would require constants k_2 and k_3, not both zero, such that

$$\vec{\mathbf{v}}_1 = k_2\vec{\mathbf{v}}_2 + k_3\vec{\mathbf{v}}_3 \iff \vec{\mathbf{v}}_1 - k_2\vec{\mathbf{v}}_2 - k_3\vec{\mathbf{v}}_3 = \vec{\mathbf{0}}$$

which implies that the vectors are linearly dependent, contradicting the original statement that the vectors are linearly independent.

ii. Let $\vec{\mathbf{x}}_1 = \begin{bmatrix} a_1 \\ b_1 \\ 0 \\ a_1b_1 \end{bmatrix}$ and $\vec{\mathbf{x}}_2 = \begin{bmatrix} a_2 \\ b_2 \\ 0 \\ a_2b_2 \end{bmatrix}$ be vectors in $\mathbb{W}$ and c, d be scalars. Then (only one of these methods is necessary)

Method 1 (check closure with respect to linear combinations)

$$c\vec{\mathbf{x}}_1 + d\vec{\mathbf{x}}_2 = c\begin{bmatrix} a_1 \\ b_1 \\ 0 \\ a_1b_1 \end{bmatrix} + d\begin{bmatrix} a_2 \\ b_2 \\ 0 \\ a_2b_2 \end{bmatrix} = \begin{bmatrix} ca_1 + da_2 \\ cb_1 + db_2 \\ 0 \\ ca_1b_1 + da_2b_2 \end{bmatrix} \neq \begin{bmatrix} ca_1 + da_2 \\ cb_1 + db_2 \\ 0 \\ (ca_1 + da_2)(cb_1 + db_2) \end{bmatrix}$$

Thus $c\vec{\mathbf{x}}_1 + d\vec{\mathbf{x}}_2$ is not in $\mathbb{W}$ meaning $\mathbb{W}$ is not closed with respect to linear combinations and consequently cannot be a subspace.

Method 2 (check closure with respect to scalar multiplication)

$$c\vec{\mathbf{x}}_1 = c\begin{bmatrix} a_1 \\ b_1 \\ 0 \\ a_1b_1 \end{bmatrix} = \begin{bmatrix} ca_1 \\ cb_1 \\ 0 \\ ca_1b_1 \end{bmatrix} \neq \begin{bmatrix} ca_1 \\ cb_1 \\ 0 \\ (ca_1)(cb_1) \end{bmatrix}$$

Thus $c\vec{\mathbf{x}}_1$ is not in $\mathbb{W}$ meaning $\mathbb{W}$ is not closed with respect to scalar multiplication and consequently cannot be a subspace.

Method 3 (check closure with respect to vector addition)

$$\vec{\mathbf{x}}_1 + \vec{\mathbf{x}}_2 = \begin{bmatrix} a_1 \\ b_1 \\ 0 \\ a_1b_1 \end{bmatrix} + \begin{bmatrix} a_2 \\ b_2 \\ 0 \\ a_2b_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 \\ b_1 + b_2 \\ 0 \\ a_1b_1 + a_2b_2 \end{bmatrix} \neq \begin{bmatrix} a_1 + a_2 \\ b_1 + b_2 \\ 0 \\ (a_1 + a_2)(b_1 + b_2) \end{bmatrix}$$

Thus $\vec{\mathbf{x}}_1 + \vec{\mathbf{x}}_2$ is not in $\mathbb{W}$ meaning $\mathbb{W}$ is not closed with respect to vector addition and consequently cannot be a subspace.

(b) There are 3 vectors in a space with dimension 3 so we check for linear independence.

Method 1 Using the Wronskian.

$$W\left[(t-2)^2, t^2-2, -2\right](t) = \begin{vmatrix} (t-2)^2 & t^2-2 & -2 \\ 2(t-2) & 2t & 0 \\ 2 & 2 & 0 \end{vmatrix} = -2(-1)^{1+3}\begin{vmatrix} 2(t-2) & 2t \\ 2 & 2 \end{vmatrix} = -8 \neq 0$$

The vectors are linearly independent and thus form a basis for $\mathbb{P}_2$.

Mehthod 2 Using the definition.

$$c_1(t-2)^2 + c_2(t^2-2) + c_3(-2) = 0$$

$$c_1(t^2 - 4t + 4) + c_2(t^2 - 2) - 2c_3 = 0$$

$$(c_1 + c_2)t^2 - 4c_1t + (4c_1 - 2c_2 - 2c_3) = 0t^2 + 0t + 0$$

Equating like powers, this yields the linear system

$$\begin{aligned} c_1 + c_2 &= 0 \\ -4c_1 &= 0 \\ 4c_1 - 2c_2 - 2c_3 &= 0 \end{aligned}$$

By inspection, the only solution to this system is $c_1 = c_2 = c_3 = 0$, implying that the functions are linearly independent. Alternatively, writing the system using matrices gives

$$\begin{bmatrix} 1 & 1 & 0 \\ -4 & 0 & 0 \\ 4 & -2 & -2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The determinant of the coefficient matrix is -8 implying that the system has the unique solution $c_1 = c_2 = c_3 = 0$.

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