- This exam is worth 100 points and has 5 questions.
- **Show all work and simplify your answers!** Answers with no justification will receive no points unless otherwise noted.
- **Please begin each problem on a new page.**
- **DO NOT** leave the exam until you have satisfactorily scanned and uploaded your exam to Gradescope.
- You are taking this exam in a proctored and honor code enforced environment. **NO** calculators, cell phones, or other electronic devices or the internet are permitted. You are allowed one 8.5"× 11" crib sheet with writing on one side.

At the top of the page containing your answer to problem 1, write the following italicized statement and sign your name to it: "*I will abide by the CU Boulder Honor Code on this exam.*" **FAILURE TO INCLUDE THIS MAY RESULT IN A PENALTY.**

1. [2360/062422 (14 pts)] Consider the linear system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ where

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 1 \\ -2 & 1 & 0 \\ 2 & -3 & 4 \end{bmatrix}, \quad \vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \vec{\mathbf{b}} = \begin{bmatrix} 2 \\ -2 \\ 6 \end{bmatrix}$$

(a) (6 pts) Calculate $|\mathbf{A}|$ using the cofactor expansion method.

(b) (2 pts) Is there a unique solution to $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$? Justify your answer.

(c) (6 pts) The RREF of the augmented matrix for the system is $\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$. Find the solution to the original system, writing your answer in the form $\vec{\mathbf{x}} = \vec{\mathbf{x}}_h + \vec{\mathbf{x}}_p$, clearly labeling $\vec{\mathbf{x}}_h$ and $\vec{\mathbf{x}}_p$.

2. [2360/062422 (12 pts)] Solve the following linear system by finding the inverse of an appropriate matrix.

$$\begin{aligned} x_1 + x_3 &= 2 \\ 2x_1 - 4x_2 + 6x_3 &= 4 \\ 3x_1 + 2x_2 - x_3 &= 4 \end{aligned}$$

3. [2360/062422 (30 pts)] The following parts are not related.

(a) (6 pts) Consider the matrix $\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ 2 & 1 & -5 \\ 6 & -4 & k \end{bmatrix}$. Find the value of k, if any, such that $\vec{\mathbf{x}} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$ is an eigenvector of $\mathbf{A}$ associated with the eigenvalue 5. Do not find any other eigenvalues or eigenvectors.

(b) (9 pts) Consider the matrix $\mathbf{A} = \begin{bmatrix} (b-1) & 0 & 0 \\ 7 & (b+1) & 0 \\ -3 & 2 & b^2 \end{bmatrix}$.

i. (3 pts) Find all values of b such that the matrix has 0 as an eigenvalue.

ii. (3 pts) What is $|\mathbf{A}|$ for the values of b found in part (i)?

iii. (3 pts) For the values of b found in part (i), is the system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{0}}$, where $\vec{\mathbf{x}} \in \mathbb{R}^3$, consistent? Explain briefly.

(c) (5 pts) The characteristic polynomial for a certain matrix is $p(\lambda) = \lambda^3 (\lambda + 1)^2 (\lambda^2 + 4)$.

i. (4 pts) Find the eigenvalues of the matrix and state the multiplicity of each.

ii. (1 pt) What is the order/size of the matrix from which the characteristic equation was derived?

(d) (10 pts) Let $\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$.

i. (8 pts) Find a basis for the eigenspace associated with the real eigenvalue of $\mathbf{A}$.

ii. (2 pts) What is the dimension of the eigenspace in part (i)?

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4. [2360/062422 (24 pts)] Let $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$. Evaluate each of the following expressions or explain why it is not defined.

(a) $\mathbf{AB}$ (b) $\mathbf{B} + 2\mathbf{I}$ (c) $\left(\mathbf{A}^{\mathrm{T}}\mathbf{A}\right)^{\mathrm{T}}$ (d) $|\mathbf{A}|\mathbf{A}^{-1}$ (e) $\mathbf{B}^{\mathrm{T}}\mathbf{A}$ (f) $\operatorname{Tr}\left(\mathbf{B}^2\right)$

5. [2360/062422 (20 pts)] The following parts are not related. However, you need to provide justification for all of your answers for each part. Correct answers with missing or incorrect justifications will receive no points.

(a) (14 pts) Consider the vector space $\mathbb{R}^4$.

i. (8 pts) Suppose $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$ are vectors in $\mathbb{R}^4$ and that the only solution to $c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + c_3\vec{\mathbf{v}}_3 = \vec{\mathbf{0}}$ is $c_1 = c_2 = c_3 = 0$.

A. (4 pts) Is span $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\} = \mathbb{R}^4$?

B. (4 pts) Is $\vec{\mathbf{v}}_1 \in$ span $\{\vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$?

ii. (6 pts) Let $\mathbb{W}$ be the set of vectors $\mathbb{R}^4$ of the form $\begin{bmatrix} a \\ b \\ 0 \\ ab \end{bmatrix}$ where a and b are real numbers. Is $\mathbb{W}$ a subspace of $\mathbb{R}^4$?

(b) (6 pts) Can the set $\left\{(t-2)^2, t^2 - 2, -2\right\}$ be a basis for $\mathbb{P}_2$, the vector space of all polynomials of degree less than or equal to 2?