1. [2360/061022 (24 pts)] A 200 liter tank initially contains 100 liters (L) of pure water. Water enters the tank at a rate of 2 L/hr and the water entering the tank has a kool-aid concentration of 2 gram/L. If a well-mixed solution leaves the tank at a rate of 1 L/hr, how much kool-aid powder is dissolved in the tank when it overflows? Be sure to define the variables you use and use the integrating factor method.

   SOLUTION:

   Since the flow into the tank differs from the flow out, the volume, V, of solution in the tank is a function of t.

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \text{flow in} - \text{flow out} = 2 - 1 = 1, \quad V(0) = 100$$

$$V(t) = t + C$$

$$V(0) = 100 = 0 + C \implies C = 100$$

$$V(t) = t + 100$$

   Let $x(t)$ be the amount (grams) of kool-aid solution in the tank at time t. Then

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \text{mass rate in} - \text{mass rate out} = \left(2\,\frac{\text{gram}}{\text{L}}\right)\left(2\,\frac{\text{L}}{\text{hr}}\right) - \left(\frac{x}{t+100}\,\frac{\text{grams}}{\text{L}}\right)\left(1\,\frac{\text{L}}{\text{hr}}\right)$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{x}{t+100} = 4$$

   Integrating factor

$$\int \frac{\mathrm{d}t}{t+100} = \ln|t+100| \implies \mu(t) = e^{\ln|t+100|} = t + 100 \quad \text{(absolute values not needed since } t + 100 > 0)$$

   Then, multiplying by the integrating factor gives

$$[(t+100)x]' = 4(t+100)$$

$$(t+100)x = 4\int (t+100)\,\mathrm{d}t = 2(t+100)^2 + C$$

$$x(t) = 2(t+100) + \frac{C}{t+100}$$

   Since the tank initially contains pure water, $x(0) = 0$. Applying this initial condition yields

$$x(0) = 2(100) + \frac{C}{100} = 0 \implies C = -20000$$

   giving the amount of dissolved kool-aid in the tank at time t as

$$x(t) = 2(t+100) - \frac{20000}{t+100}$$

   The tank fills when the volume is 200 L, which occurs at $t = 100$ hours. Thus

$$x(100) = 2(100+100) - \frac{20000}{100+100} = 400 - 100 = 300\,\text{grams}$$

2. [2360/061022 (16 pts)] Find the general solution of $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2 + \sqrt{y - 2x + 3}$ by using the substitution $u = y - 2x + 3$. Write your solution as an explicit function (be sure to simplify).

   SOLUTION:

The substitution $u = y - 2x + 3$ gives $\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}x} - 2$ so that the differential equation becomes separable

$$\frac{\mathrm{d}u}{\mathrm{d}x} + 2 = 2 + \sqrt{u}$$
$$\int u^{-1/2}\,\mathrm{d}u = \int \mathrm{d}x$$
$$2\sqrt{u} = x + C$$
$$2\sqrt{y - 2x + 3} = x + C$$
$$4(y - 2x + 3) = (x + C)^2$$
$$y = \frac{1}{4}(x + C)^2 + 2x - 3$$

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3. [2360/061022 (18 pts)] The following parts are not related.

(a) (6 pts) Consider the system of differential equations

$$\begin{aligned} x' &= y^4 + 1 \\ y' &= x + 1 \end{aligned}$$

i. Find the h and v nullclines of the system.
ii. Find the equilibrium points, if any exist.

(b) (12 pts) Consider the initial value problem $(x + 1)w'(x) + w = 2(x + 1)$, $w(1) = 5$, $x \geq 0$. The solution to the associated homogeneous equation is $w_h(x) = c(x + 1)^{-1}$.

i. Complete the Euler-Lagrange two stage method (variation of parameters) to find the particular solution.
ii. Solve the initial value problem.

SOLUTION:

(a) i. The h nullclines occur where $y' = x + 1 = 0$. Thus the line $x = -1$ is the h nullcline. The v nullclines occur where $x' = y^4 + 1 = 0$. This has no solution in the real numbers. There are no v nullclines.

ii. Equilibrium points occur at the intersection of the h and v nullclines. Since there are no v nullclines, there can be no intersections of h and v nullclines and consequently no equilibrium points.

(b) i. The result of stage one of the method is given as $w_h(x) = c(x + 1)^{-1}$.

$$w_p = v(x)(x + 1)^{-1}$$
$$w_p' + \frac{1}{x + 1}w_p = -v(x)(x + 1)^{-2} + v'(x)(x + 1)^{-1} + \frac{1}{x + 1}v(x)(x + 1)^{-1} = 2$$
$$v'(x) = 2(x + 1)$$
$$v(x) = (x + 1)^2$$
$$w_p(x) = x + 1$$

ii. Using the Nonhomogeneous Principle, the general solution is $w = w_h + w_p = c(x + 1)^{-1} + x + 1$. Applying the initial condition gives $w(1) = \frac{c}{2} + 1 + 1 = 5 \implies c = 6$. The solution to the initial value problem is

$$w(x) = \frac{6}{x + 1} + x + 1 = \frac{x^2 + 2x + 7}{x + 1}$$

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4. [2360/061022 (16 pts)] The following parts are not related.

(a) (8 pts) Consider the differential equation $y' = (y - 1)^{2/3}e^{2t}\tan 2t$. What conclusions, if any, can be drawn from Picard's theorem regarding the existence and uniqueness of solutions to the initial value problems consisting of the differential equation and the following initial conditions:

i. $y(0) = 1$ ii. $y(\pi/2) = 0$

(b) (8 pts) Use one step of Euler's method to approximate the solution of $y' = 2\pi t \sin y,\ y(1) = \pi/2$ at $t = 1.5$.

SOLUTION:

(a) $f(t,y) = (y-1)^{2/3}e^{2t}\tan 2t$ which is continuous except at odd integer multiples of $\pi/4$. $f_y(t,y) = \frac{2}{3}(y-1)^{-1/3}e^{2t}\tan 2t$ which is continuous except at odd integer multiples of $\pi/4$ and if $y \neq 1$.

i. Picard's theorem guarantees the existence of a solution to the IVP on some interval since $f(t,y)$ is continuous in a neighborhood (rectangle) containing $(t,y) = (0,1)$. Since $f_y(t,y)$ is not continuous at $(t,y) = (0,1)$, Picard's theorem tells us nothing about the uniqueness of solutions to the IVP.

ii. Picard's theorem guarantees the existence of a unique solution to the IVP on some interval since $f(t,y)$ and $f_y(t,y)$ are both continuous in a neighborhood of $(\pi/2, 0)$.

(b) We have $h = 0.5, t_0 = 1, y_0 = y(1) = \pi/2$. Thus Euler's method yields

$$y(1.5) \approx y_1 = y_0 + hf(t_0,y_0) = y_0 + h(2\pi)t_0\sin(y_0) = \pi/2 + 0.5(2\pi)(1)\sin(\pi/2) = 3\pi/2$$

5. [2360/061022 (26 pts)] The following parts are not related.

(a) (16 pts) Consider the differential equation $x'(t) = x^2(x^2+1)(x^2+x-6)$.

i. (6 pts) Find all the equilibrium solutions and determine their stability.

ii. (7 pts) Draw the phase line of the equation.

iii. (3 pts) Let $x_1(t)$ be the solution that passes through the point $(0,-2)$. What is $\lim\limits_{t\to\infty} x_1(t)$?

(b) (10 pts) On your paper, write the letters A-E in a column. Next to each letter, write the order of each differential equation followed by the numbers (I-III) corresponding to all the classes to which the differential equation belongs. No justification required and no partial credit available. (I) separable (II) linear (III) homogeneous

A. $y' - 1 = 0$ B. $\dfrac{dy}{dx} = \sqrt{1 + \left(\dfrac{d^2y}{dx^2}\right)^2}$ C. $y''' + 6y'' - 8y = \sin t$

D. $(\sin t)x'' = x'$ E. $e^y y' - x\tan x = 0$

SOLUTION:

(a) The equation can be written as $x' = x^2(x^2+1)(x-2)(x+3)$.

i. The equilibrium solutions are the values of x that make the right hand side vanish, that is $x = -3, 0, 2$ are the equilibrium solutions. Evaluating the right hand side at points other than the equilibrium solutions we have

$$\begin{aligned} x > 2: &\quad x' > 0 \\ 2 > x > 0: &\quad x' < 0 \\ 0 > x > -3: &\quad x' < 0 \\ -3 > x: &\quad x' > 0 \end{aligned}$$

Thus $x = -3$ is stable, $x = 0$ is semistable, and $x = 2$ is unstable.

ii. Phase line in the following figure.

$$\begin{array}{c} \uparrow \\ 2 \ \circ \\ \downarrow \\ 0 \ ◓ \\ \downarrow \\ -3 \ \bullet \\ \uparrow \end{array}$$

iii. Looking at the phase line, for the given initial value $x'(t) < 0$ so $x_1(t)$ is a decreasing function and

$$\lim_{t\to\infty} x_1(t) = -3$$

(b) A. first order; I, II B. second order; none C. third order; II D. second order; I, II, III E. first order; I

Note: As written, the equation in D is not separable. However, using the substitution $y = x'$ it becomes separable.