1. [2360/072321 Exam (14 pts)] Let $L(y)$ represent a linear operator describing a fourth order differential equation. Consider the set of solutions to $L(y) = 0$ given by $\{t-1, 2t+1, t^2-7t+3, 4t^2+8t\}$. Does this set constitute a basis for the solution space of $L(y) = 0$? Justify your answer completely.

SOLUTION:

The dimension of the solution space is four so if this set is linearly independent, then it will form a basis.

$$W[t-1, 2t+1, t^2-7t+3, 4t^2+8t](t) = \begin{vmatrix} t-1 & 2t+1 & t^2-7t+3 & 4t^2+8t \\ 1 & 2 & 2t-7 & 8t+8 \\ 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0 \quad \text{(note row of zeros)}$$

Since the Wronksian vanishes identically and the functions are solutions to a differential equation, they are linearly dependent and thus cannot form a basis for the solution space. ■

2. [2360/072321 Exam (20 pts)] A critically damped, unforced harmonic oscillator consisting of a one-quarter kilogram mass and a spring with a restoring constant of 25 newtons per meter is oriented horizontally. Let $x(t)$ be the position of the mass at time t.

   (a) (4 pts) Write the differential equation governing the motion of the oscillator.

   (b) (4 pts) Suppose the motion is started when $t = 0$ by pushing the mass to the left at 10 meters per second from a position 7 meters to the right of the rest position. What are the initial conditions?

   (c) (2 pts) If the oscillator were undamped and driven by the function $f(t) = \cos \omega_f t$, what value of ω_f would result in unbounded solutions to the initial value problem?

   (d) (10 pts) Now suppose an external driving force, $f(t)$, is applied to the oscillator as follows: There is no driving force for the first 5 seconds. At 5 seconds, a driving force of $t-5$ is applied. Five seconds later the driving force is $5e^{-(t-10)}$. Finally, at 20 seconds, a unit impulse is applied. Write this driving force as a single function (not piecewise).

SOLUTION:

   (a) We have $m = 1/4, k = 25$ and since the oscillator is critically damped, $b^2 - 4\left(\frac{1}{4}\right)(25) = 0 \implies b = 5$. The differential equation is thus

$$\frac{1}{4}\ddot{x} + 5\dot{x} + 25x = 0$$

   (b) $x(0) = 7, \dot{x}(0) = -10$

   (c) Unbounded solutions are the result of the oscillator being in resonance.

$$\omega_f = \sqrt{\frac{25}{1/4}} = 10$$

   (d) $f(t) = (t-5)\text{step}(t-5) - (t-5)\text{step}(t-10) + 5e^{-(t-10)}\text{step}(t-10) + \delta(t-20)$

■

3. [2360/072321 Exam (40 pts)] Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}$.

   (a) (10 pts) One of the eigenvalues of $\mathbf{A}$ is $\lambda = 1$. Calculate $\mathbf{A}\vec{\mathbf{x}}$ where $\vec{\mathbf{x}} = \begin{bmatrix} -1 & 4 & 1 \end{bmatrix}^{\mathrm{T}}$. What can you say about $\vec{\mathbf{x}}$?

   (b) (10 pts) Another eigenvalue of $\mathbf{A}$ is $\lambda = 3$. Suppose after some elementary row operations on the augmented matrix associated with the linear system $(\mathbf{A} - 3\mathbf{I})\vec{\mathbf{v}} = \vec{\mathbf{0}}$ you obtain

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 5 & 0 & -5 & 0 \\ 2 & 1 & -4 & 0 \end{array}\right]$$

      i. (5 pts) Put this matrix into RREF.

      ii. (5 pts) Find a basis for the eigenspace associated with $\lambda = 3$. What is its dimension?

   (c) (10 pts) The third eigenvalue of $\mathbf{A}$ is $\lambda = -2$ with associated eigenvector $\vec{\mathbf{v}} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$. Write the general solution of the system of differential equations $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}}$.

(d) (10 pts) Do the columns of $\mathbf{A}$ form a basis for $\mathbb{R}^3$? Justify your answer.

SOLUTION:

(a)

$$\mathbf{A}\vec{\mathbf{x}} = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} = 1\vec{\mathbf{x}} \implies \begin{bmatrix} -1 & 4 & 1 \end{bmatrix}^{\mathrm{T}} \quad \text{is the eigenvector associated with the eigenvalue 1}$$

(b) i.

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 5 & 0 & -5 & 0 \\ 2 & 1 & -4 & 0 \end{array}\right] \xrightarrow{R_2^* = \frac{1}{5}R_2} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 2 & 1 & -4 & 0 \end{array}\right] \xrightarrow{R_3^* = -2R_2 + R_3} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2; R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

ii. From the RREF, $v_1 = v_3, v_2 = 2v_3$ with v_3 the free variable. A basis for the eigenspace is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$ which has a dimension of 1.

(c)

$$\vec{\mathbf{x}}(t) = c_1 e^t \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + c_3 e^{-2t} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

(d) There are three vectors given and the dimension of $\mathbb{R}^3$ is three so we only need to check that the vectors are linearly independent. To this end, we need to show that the only solution of

$$c_1 \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 4 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

is $c_1 = c_2 = c_3 = 0$. This is equivalent to the linear system

$$\begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

with

$$\begin{vmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{vmatrix} = -6$$

showing that the system has only the trivial solution $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. This proves that the vectors are linearly independent implying that the columns of $\mathbf{A}$ do form a basis for $\mathbb{R}^3$.

■

4. [2360/072321 Exam (30 pts)] Use Laplace transforms to solve the initial value problem

$$y'' + 25y = 25[1 - \text{step}(t-4)],\; y(0) = y'(0) = 0$$

SOLUTION:

Taking Laplace transforms of both sides yields

$$s^2 Y(s) - sy(0) - y'(0) + 25Y(s) = 25\left(\frac{1 - e^{-4s}}{s}\right)$$

$$Y(s) = 25\left(\frac{1 - e^{-4s}}{s(s^2+25)}\right)$$

$$\frac{25}{s(s^2+25)} = \frac{A}{s} + \frac{Bs + C}{s^2+25} \text{ after some algebra} = \frac{1}{s} - \frac{s}{s^2+25}$$

$$Y(s) = \frac{1}{s} - \frac{s}{s^2+25} - e^{-4s}\left(\frac{1}{s} - \frac{s}{s^2-25}\right)$$

$$y(t) = \mathscr{L}^{-1}\{Y(s)\} = 1 - \cos 5t - [1 - \cos 5(t-4)]\,\text{step}(t-4)$$

■

5. [2360/072321 Exam (20 pts)] Solve the initial value problem $\vec{\mathbf{x}}' = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \vec{\mathbf{x}}$, $\vec{\mathbf{x}}(0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ knowing that one eigenvalue/eigenvector pair is $\lambda = 1 + i, \vec{\mathbf{v}} = \begin{bmatrix} i \\ 1 \end{bmatrix}$.

SOLUTION:

We have $\alpha = 1, \beta = 1, \mathbf{p} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{q} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Thus the general solution is

$$\begin{aligned} \vec{\mathbf{x}}(t) &= c_1 \left(e^t \cos t \begin{bmatrix} 0 \\ 1 \end{bmatrix} - e^t \sin t \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) + c_2 \left(e^t \sin t \begin{bmatrix} 0 \\ 1 \end{bmatrix} + e^t \cos t \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \\ &= \begin{bmatrix} -c_1 e^t \sin t + c_2 e^t \cos t \\ c_1 e^t \cos t + c_2 e^t \sin t \end{bmatrix} \end{aligned}$$

Applying the initial conditions yields

$$\begin{aligned} 0 + c_2 &= -1 \\ c_1 + 0 &= 1 \end{aligned}$$

giving the unique solution to the system of differential equations as

$$\vec{\mathbf{x}}(t) = e^t \begin{bmatrix} -\sin t - \cos t \\ \cos t - \sin t \end{bmatrix}$$

■

6. [2360/072321 Exam (26 pts)] Make a legible table on your paper corresponding to the questions below and write the word **TRUE** or **FALSE** in the appropriate place in your table. No partial credit will be awarded and no work is required to be shown.

(a) Consider the linear system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ where $\mathbf{A}$ is an $n \times n$ matrix.

  i. If $|\mathbf{A}| = 0$, and $\vec{\mathbf{b}} \neq \vec{\mathbf{0}}$ the linear system always has infinitely many solutions.

  ii. If $\mathbf{A}$ has zero as an eigenvalue, the linear system with $\vec{\mathbf{b}} = \vec{\mathbf{0}}$ is consistent.

  iii. The solution to the linear system is always $\vec{\mathbf{x}} = \mathbf{A}^{-1}\vec{\mathbf{b}}$.

  iv. The solution space to the system where $\vec{\mathbf{b}}$ is an $n \times 1$ column vector of ones is a subspace of $\mathbb{R}^n$.

(b) Consider the system of differential equations $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}}$ where $\mathbf{A}$ is a 2×2 matrix.

  i. If Tr$\mathbf{A} = 0$ and $|\mathbf{A}| \neq 0$, the fixed point at $(0, 0)$ is a stable node.

  ii. If $(\text{Tr}\mathbf{A})^2 - 4|\mathbf{A}| < 0$ and Tr$\mathbf{A} < 0$, then all solutions of the system will approach 0 as $t \to \infty$.

  iii. If $|\mathbf{A}| \neq 0$, the fixed point at $(0, 0)$ in the phase plane is always stable.

  iv. The v and h nullclines are lines.

  v. If Tr$\mathbf{A} = 10$ and $|\mathbf{A}| = 25$ the equilibrium solution is an unstable degenerate node.

(c) Consider the differential equation $y' = 3t^2y^2$.

  i. Picard's Theorem guarantees the existence of a unique solution for any initial condition $y(t_0) = y_0$.

  ii. The unique solution to the initial value problem consisting of the differential equation and the initial condition $y(0) = 1$ is $y = -(t^3 - 1)^{-1}$.

  iii. The equation is a second order linear homogeneous equation.

  iv. Euler's method cannot be used to approximate solutions to the differential equation that pass through the origin.

SOLUTION:

(a) i. **FALSE** - the system may be inconsistent, in which case there are no solutions

  ii. **TRUE** - homogeneous linear systems are always consistent

  iii. **FALSE** - the inverse matrix may not exist

  iv. **FALSE** - the system is nonhomogeneous and the solution space does not contain the zero vector

(b) i. **FALSE** - it is either a center or saddle

  ii. **TRUE** - this puts us above the parabola in the second quadrant

  iii. **FALSE** - for example, if $|\mathbf{A}| < 0$ fixed point is a saddle which is unstable

  iv. **TRUE** - the system has the form $x_1' = ax_1 + bx_2$ and $x_2' = cx_1 + dx_2$. Setting these to 0 results in lines.

  v. **FALSE** - $(\text{Tr}\mathbf{A})^2 - 4|\mathbf{A}| = 0, \text{Tr}\mathbf{A} > 0$; could be an unstable degenerate node or star node

(c) i. **TRUE** - $f(t,y) = t^2y^2$ and $f_y(t,y) = 2t^2y$ are continuous for all t and y

ii. **TRUE** - plug the solution into the differential equation and see that an identity results; check that the initial condition is satisfied

iii. **FALSE** - it is first order and nonlinear

iv. **FALSE** - Euler's method is always applicable to any differential equation $y' = f(t,y)$

■

Short table of Laplace Transforms: $\mathscr{L}\{f(t)\} = F(s) \equiv \int_0^\infty e^{-st} f(t)\,\mathrm{d}t$

In this table, a, b, c are real numbers with $c \geq 0$, and $n = 0, 1, 2, 3, \ldots$

$$\mathscr{L}\left\{t^n e^{at}\right\} = \frac{n!}{(s-a)^{n+1}} \qquad \mathscr{L}\left\{e^{at}\cos bt\right\} = \frac{s-a}{(s-a)^2+b^2} \qquad \mathscr{L}\left\{e^{at}\sin bt\right\} = \frac{b}{(s-a)^2+b^2}$$

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{\mathrm{d}^n F(s)}{\mathrm{d}s^n} \qquad \mathscr{L}\left\{e^{at} f(t)\right\} = F(s-a) \qquad \mathscr{L}\{\delta(t-c)\} = e^{-cs}$$

$$\mathscr{L}\{t f'(t)\} = -F(s) - s\frac{\mathrm{d}F(s)}{\mathrm{d}s} \qquad \mathscr{L}\{f(t-c)\,\text{step}(t-c)\} = e^{-cs}F(s) \qquad \mathscr{L}\{f(t)\,\text{step}(t-c)\} = e^{-cs}\mathscr{L}\{f(t+c)\}$$

$$\mathscr{L}\left\{f^{(n)}(t)\right\} = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \cdots - f^{(n-1)}(0)$$
