1. [APPM 2360/072321 Exam (14 pts)] Let $L(y)$ represent a linear operator describing a fourth order differential equation. Consider the set of solutions to $L(y) = 0$ given by \{ $t - 1, 2t + 1, t^2 - 7t + 3, 4t^2 + 8t$ \}. Does this set constitute a basis for the solution space of $L(y) = 0$? Justify your answer completely.

**Solution:**

The dimension of the solution space is four so if this set is linearly independent, then it will form a basis.

$$W[t - 1, 2t + 1, t^2 - 7t + 3, 4t^2 + 8t](t) = \begin{vmatrix} 1 & 2t + 1 & t^2 - 7t + 3 & 4t^2 + 8t \\ t - 1 & 2 & 2t - 7 & 8t \\ 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0 \quad \text{(note row of zeros)}$$

Since the Wronskian vanishes identically and the functions are solutions to a differential equation, they are linearly dependent and thus cannot form a basis for the solution space.

2. [APPM 2360/072321 Exam (20 pts)] A critically damped, unforced harmonic oscillator consisting of a one-quarter kilogram mass and a spring with a restoring constant of 25 newtons per meter is oriented horizontally. Let $x(t)$ be the position of the mass at time $t$.

(a) (4 pts) Write the differential equation governing the motion of the oscillator.

(b) (4 pts) Suppose the motion is started when $t = 0$ by pushing the mass to the left at 10 meters per second from a position 7 meters to the right of the rest position. What are the initial conditions?

(c) (2 pts) If the oscillator were undamped and driven by the function $f(t) = \cos \omega_f t$, what value of $\omega_f$ would result in unbounded solutions to the initial value problem?

(d) (10 pts) Now suppose an external driving force, $f(t)$, is applied to the oscillator as follows: There is no driving force for the first 5 seconds. At 5 seconds, a driving force of $t - 5$ is applied. Five seconds later the driving force is $5e^{-(t-10)}$. Finally, at 20 seconds, a unit impulse is applied. Write this driving force as a single function (not piecewise).

**Solution:**

(a) We have $m = 1/4$, $k = 25$ and since the oscillator is critically damped, $b^2 - 4 \left( \frac{1}{4} \right) (25) = 0 \implies b = 5$. The differential equation is thus

$$\frac{1}{4} \ddot{x} + 5\dot{x} + 25x = 0$$

(b) $x(0) = 7$, $\dot{x}(0) = -10$

(c) Unbounded solutions are the result of the oscillator being in resonance.

$$\omega_f = \sqrt{\frac{25}{1/4}} = 10$$

(d) $f(t) = (t - 5)\text{step}(t - 5) - (t - 5)\text{step}(t - 10) + 5e^{-(t-10)}\text{step}(t - 10) + \delta(t - 20)$

3. [APPM 2360/072321 Exam (40 pts)] Consider the matrix $A = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}$.

(a) (10 pts) One of the eigenvalues of $A$ is $\lambda = 1$. Calculate $A \vec{x}$ where $\vec{x} = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}^T$. What can you say about $\vec{x}$?

(b) (10 pts) Another eigenvalue of $A$ is $\lambda = 3$. Suppose after some elementary row operations on the augmented matrix associated with the linear system $(A - 3I) \vec{v} = \vec{0}$ you obtain

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 5 & 0 & -5 & 0 \\ 2 & 1 & -4 & 0 \end{bmatrix}$$

i. (5 pts) Put this matrix into RREF.

ii. (5 pts) Find a basis for the eigenspace associated with $\lambda = 3$. What is its dimension?

(c) (10 pts) The third eigenvalue of $A$ is $\lambda = -2$ with associated eigenvector $\vec{v} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$. Write the general solution of the system of differential equations $\vec{x}' = A \vec{x}$. 


(d) (10 pts) Do the columns of \( A \) form a basis for \( \mathbb{R}^3 \)? Justify your answer.

**SOLUTION:**

(a)

\[
A \vec{x} = \begin{bmatrix}
1 & -1 & 4 \\
3 & 2 & -1 \\
2 & 1 & -1
\end{bmatrix} \begin{bmatrix}
-1 \\
4 \\
1
\end{bmatrix} = \begin{bmatrix}
-1 \\
4 \\
1
\end{bmatrix} = 1 \vec{x} \implies \begin{bmatrix}
-1 \\
4 \\
1
\end{bmatrix}^T \text{ is the eigenvector associated with the eigenvalue 1}
\]

(b) i.

\[
\begin{bmatrix}
0 & 0 & 0 \\
5 & -5 & 0 \\
2 & 1 & -4
\end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
2 & 1 & 4
\end{bmatrix} \xrightarrow{R_3 = -2R_2 + R_4} \begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 1 & -2
\end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix}
1 & 0 & -1 \\
0 & 1 & -2 \\
0 & 0 & 0
\end{bmatrix}
\]

ii. From the RREF, \( v_1 = v_3, v_2 = 2v_3 \) with \( v_3 \) the free variable. A basis for the eigenspace is \( \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \) which has a dimension of 1.

(c)

\[
\vec{x}(t) = c_1e^t \begin{bmatrix}
-1 \\
4 \\
1
\end{bmatrix} + c_2e^{3t} \begin{bmatrix}
1 \\
2 \\
1
\end{bmatrix} + c_3e^{-2t} \begin{bmatrix}
1 \\
-1 \\
-1
\end{bmatrix}
\]

(d) There are three vectors given and the dimension of \( \mathbb{R}^3 \) is three so we only need to check that the vectors are linearly independent. To this end, we need to show that the only solution of

\[
\begin{bmatrix}
1 & 3 \\
2 & 2 \\
3 & 1
\end{bmatrix} \begin{bmatrix}
c_1 \\
c_2 \\
c_3
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]

is \( c_1 = c_2 = c_3 = 0 \). This is equivalent to the linear system

\[
\begin{bmatrix}
1 & -1 & 4 \\
3 & 2 & -1 \\
2 & 1 & -1
\end{bmatrix} \begin{bmatrix}
c_1 \\
c_2 \\
c_3
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]

with

\[
\begin{bmatrix}
1 & -1 & 4 \\
3 & 2 & -1 \\
2 & 1 & -1
\end{bmatrix} = -6
\]

showing that the system has only the trivial solution \( \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \). This proves that the vectors are linearly independent implying that the columns of \( A \) do form a basis for \( \mathbb{R}^3 \).

4. [2360/072321 Exam (30 pts)] Use Laplace transforms to solve the initial value problem

\[
y'' + 25y = 25[1 - \text{step}(t - 4)], \ y(0) = y'(0) = 0
\]

**SOLUTION:**

Taking Laplace transforms of both sides yields

\[
s^2Y(s) - sy(0) - y'(0) + 25Y(s) = 25 \left( \frac{1 - e^{-4s}}{s} \right)
\]

\[
Y(s) = 25 \left( \frac{1 - e^{-4s}}{s(s^2 + 25)} \right)
\]

\[
\frac{25}{s(s^2 + 25)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 25} \quad \text{after some algebra} \quad \frac{1}{s} - \frac{s}{s^2 + 25}
\]

\[
Y(s) = \frac{1}{s} - \frac{s}{s^2 + 25} - e^{-4s} \left( \frac{1}{s} - \frac{s}{s^2 - 25} \right)
\]

\[
y(t) = \mathcal{L}^{-1} \left\{ Y(s) \right\} = 1 - \cos 5t - [1 - \cos 5(t - 4)] \text{ step}(t - 4)
\]
5. [2360/072321 Exam (20 pts)] Solve the initial value problem \( \mathbf{x}' = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \) knowing that one eigenvalue/eigenvector pair is \( \lambda = 1 + i, \quad \mathbf{v} = \begin{bmatrix} i \\ 1 \end{bmatrix} \).

**SOLUTION:**

We have \( \alpha = 1, \beta = 1, \quad \mathbf{p} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{q} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \). Thus the general solution is

\[
\mathbf{x}(t) = c_1 \left( e^t \cos t \begin{bmatrix} 0 \\ 1 \end{bmatrix} - e^t \sin t \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) + c_2 \left( e^t \sin t \begin{bmatrix} 0 \\ 1 \end{bmatrix} + e^t \cos t \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)
\]

\[
= \begin{bmatrix} -c_1 e^t \sin t + c_2 e^t \cos t \\ c_1 e^t \cos t + c_2 e^t \sin t \end{bmatrix}
\]

Applying the initial conditions yields

\[
0 + c_2 = -1 \quad \text{and} \quad c_1 + 0 = 1
\]

giving the unique solution to the system of differential equations as

\[
\mathbf{x}(t) = e^t \begin{bmatrix} -\sin t - \cos t \\ -\cos t - \sin t \end{bmatrix}
\]

6. [2360/072321 Exam (26 pts)] Make a legible table on your paper corresponding to the questions below and write the word **TRUE** or **FALSE** in the appropriate place in your table. No partial credit will be awarded and no work is required to be shown.

(a) Consider the linear system \( \mathbf{A} \mathbf{x} = \mathbf{b} \) where \( \mathbf{A} \) is an \( n \times n \) matrix.

i. If \( |\mathbf{A}| = 0 \) and \( \mathbf{b} \neq \mathbf{0} \) the linear system always has infinitely many solutions.

ii. If \( \mathbf{A} \) has zero as an eigenvalue, the linear system with \( \mathbf{b} = \mathbf{0} \) is consistent.

iii. The solution to the linear system is always \( \mathbf{x} = \mathbf{A}^{-1} \mathbf{b} \).

iv. The solution space to the system where \( \mathbf{b} \) is an \( n \times 1 \) column vector of ones is a subspace of \( \mathbb{R}^n \).

(b) Consider the system of differential equations \( \mathbf{x}' = \mathbf{A} \mathbf{x} \) where \( \mathbf{A} \) is a \( 2 \times 2 \) matrix.

i. If \( \text{Tr} \mathbf{A} = 0 \) and \( |\mathbf{A}| \neq 0 \), the fixed point at \((0, 0)\) is a stable node.

ii. If \( (\text{Tr} \mathbf{A})^2 - 4|\mathbf{A}| < 0 \) and \( \text{Tr} \mathbf{A} < 0 \), then all solutions of the system will approach 0 as \( t \to \infty \).

iii. If \( |\mathbf{A}| \neq 0 \), the fixed point at \((0, 0)\) in the phase plane is always stable.

iv. The \( v \) and \( h \) nullclines are lines.

v. If \( \text{Tr} \mathbf{A} = 10 \) and \( |\mathbf{A}| = 25 \) the equilibrium solution is an unstable degenerate node.

(c) Consider the differential equation \( y' = 3t^2 y^2 \).

i. Picard’s Theorem guarantees the existence of a unique solution for any initial condition \( y(t_0) = y_0 \).

ii. The unique solution to the initial value problem consisting of the differential equation and the initial condition \( y(0) = 1 \) is \( y = -(t^3 - 1)^{-1} \).

iii. The equation is a second order linear homogeneous equation.

iv. Euler’s method cannot be used to approximate solutions to the differential equation that pass through the origin.

**SOLUTION:**

(a) i. **FALSE** - the system may be inconsistent, in which case there are no solutions

   ii. **TRUE** - homogeneous linear systems are always consistent

   iii. **FALSE** - the inverse matrix may not exist

   iv. **FALSE** - the system is nonhomogeneous and the solution space does not contain the zero vector

(b) i. **FALSE** - it is either a center or saddle

   ii. **TRUE** - this puts us above the parabola in the second quadrant

   iii. **FALSE** - for example, if \( |\mathbf{A}| < 0 \) fixed point is a saddle which is unstable

   iv. **TRUE** - the system has the form \( x' = ax_1 + bx_2 \) and \( x' = cx_1 + dx_2 \). Setting these to 0 results in lines.

   v. **FALSE** - \( (\text{Tr} \mathbf{A})^2 - 4|\mathbf{A}| = 0, \text{Tr} \mathbf{A} > 0 \); could be an unstable degenerate node or star node
(c)  

i. **TRUE** - $f(t, y) = t^2 y^2$ and $f_y(t, y) = 2t^2 y$ are continuous for all $t$ and $y$

ii. **TRUE** - plug the solution into the differential equation and see that an identity results; check that the initial condition is satisfied

iii. **FALSE** - it is first order and nonlinear

iv. **FALSE** - Euler’s method is always applicable to any differential equation

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**Short table of Laplace Transforms:**  
\[
\mathcal{L}\{f(t)\} = F(s) \equiv \int_0^\infty e^{-st} f(t) \, dt
\]

In this table, $a, b, c$ are real numbers with $c \geq 0$, and $n = 0, 1, 2, 3, \ldots$

\[
\begin{align*}
\mathcal{L}\{t^n e^{at}\} &= \frac{n!}{(s-a)^{n+1}} & \mathcal{L}\{e^{at}\} &= \frac{s-a}{(s-a)^2 + b^2} & \mathcal{L}\{\cos bt\} &= \frac{s-a}{(s-a)^2 + b^2} \\
\mathcal{L}\{t^n f(t)\} &= (-1)^n \frac{d^n F(s)}{ds^n} & \mathcal{L}\{\sin bt\} &= \frac{b}{(s-a)^2 + b^2} & \mathcal{L}\{\delta(t-c)\} &= e^{-cs} \\
\mathcal{L}\{t f'(t)\} &= -F(s) - s \frac{dF(s)}{ds} & \mathcal{L}\{f(t)\text{ step}(t-c)\} &= e^{-cs} F(s) & \mathcal{L}\{f(t)\text{ step}(t-c)\} &= e^{-cs} \mathcal{L}\{f(t+c)\} \\
\mathcal{L}\{f^{(n)}(t)\} &= s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - \ldots - f^{(n-1)}(0)
\end{align*}
\]