1. Let $L(y)$ represent a linear operator describing a fourth order differential equation. Consider the set of solutions to $L(y) = 0$ given by \(\{t - 1, 2t + 1, t^2 - 7t + 3, 4t^2 + 8t\}\). Does this set constitute a basis for the solution space of $L(y) = 0$? Justify your answer completely.

2. A critically damped, unforced harmonic oscillator consisting of a one-quarter kilogram mass and a spring with a restoring constant of 25 newtons per meter is oriented horizontally. Let $x(t)$ be the position of the mass at time $t$.

(a) Write the differential equation governing the motion of the oscillator.

(b) Suppose the motion is started when $t = 0$ by pushing the mass to the left at 10 meters per second from a position 7 meters to the right of the rest position. What are the initial conditions?

(c) If the oscillator were undamped and driven by the function $f(t) = \cos \omega_f t$, what value of $\omega_f$ would result in unbounded solutions to the initial value problem?

(d) Now suppose an external driving force, $f(t)$, is applied to the oscillator as follows: There is no driving force for the first 5 seconds. At 5 seconds, a driving force of $t - 5$ is applied. Five seconds later the driving force is $5e^{-(t-10)}$. Finally, at 20 seconds, a unit impulse is applied. Write this driving force as a single function (not piecewise).

3. Consider the matrix $A = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}$.

(a) One of the eigenvalues of $A$ is $\lambda = 1$. Calculate $A \vec{x}$ where $\vec{x} = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$T. What can you say about $\vec{x}$?

(b) Another eigenvalue of $A$ is $\lambda = 3$. Suppose after some elementary row operations on the augmented matrix associated with the linear system $(A - 3I) \vec{v} = \vec{0}$ you obtain $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 5 & 0 & -5 & 0 \\ 2 & 1 & -4 & 0 \end{bmatrix}$

i. Put this matrix into RREF.
ii. Find a basis for the eigenspace associated with $\lambda = 3$. What is its dimension?

(c) The third eigenvalue of $A$ is $\lambda = -2$ with associated eigenvector $\vec{v} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$. Write the general solution of the system of differential equations $\vec{x}' = A \vec{x}$.

(d) Do the columns of $A$ form a basis for $\mathbb{R}^3$? Justify your answer.

4. Use Laplace transforms to solve the initial value problem

\[ y'' + 25y = 25[1 \text{ step}(t - 4)], \quad y(0) = y'(0) = 0 \]

5. Solve the initial value problem $\vec{x}' = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \vec{x}, \quad \vec{x}(0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ knowing that one eigenvalue/eigenvector pair is $\lambda = 1 + i$, $\vec{v} = \begin{bmatrix} i \\ 1 \end{bmatrix}$.

6. Make a legible table on your paper corresponding to the questions below and write the word TRUE or FALSE in the appropriate place in your table. No partial credit will be awarded and no work is required to be shown.

(a) Consider the linear system $A \vec{x} = \vec{b}$ where $A$ is an $n \times n$ matrix.

i. If $|A| = 0$, and $\vec{b} \neq \vec{0}$ the linear system always has infinitely many solutions.

ii. If $A$ has zero as an eigenvalue, the linear system with $\vec{b} = \vec{0}$ is consistent.

iii. The solution to the linear system is always $\vec{x} = A^{-1} \vec{b}$.

iv. The solution space to the system where $\vec{b}$ is an $n \times 1$ column vector of ones is a subspace of $\mathbb{R}^n$.

(b) Consider the system of differential equations $\vec{x}' = A \vec{x}$ where $A$ is a $2 \times 2$ matrix.

i. If $\text{Tr}A = 0$ and $|A| \neq 0$, the fixed point at $(0, 0)$ is a stable node.

ii. If $(\text{Tr}A)^2 - 4|A| < 0$ and Tr$A < 0$, then all solutions of the system will approach 0 as $t \to \infty$.

iii. If $|A| \neq 0$, the fixed point at $(0, 0)$ in the phase plane is always stable.

iv. The $v$ and $h$ nullclines are lines.
v. If \( \text{Tr} A = 10 \) and \( |A| = 25 \) the equilibrium solution is an unstable degenerate node.

(c) Consider the differential equation \( y' = 3t^2 y^2 \).

i. Picard’s Theorem guarantees the existence of a unique solution for any initial condition \( y(t_0) = y_0 \).

ii. The unique solution to the initial value problem consisting of the differential equation and the initial condition \( y(0) = 1 \) is

\[
y = -\left(t^3 - 1\right)^{-1}.
\]

iii. The equation is a second order linear homogeneous equation.

iv. Euler’s method cannot be used to approximate solutions to the differential equation that pass through the origin.

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**Short table of Laplace Transforms:**

\[
\mathcal{L}\{f(t)\} = F(s) \equiv \int_0^\infty e^{-st} f(t) \, dt
\]

In this table, \( a, b, c \) are real numbers with \( c \geq 0 \), and \( n = 0, 1, 2, 3, \ldots \)

- \( \mathcal{L}\{e^{at}\} = \frac{n!}{(s-a)^{n+1}} \)
- \( \mathcal{L}\{e^{at} \cos bt\} = \frac{s-a}{(s-a)^2 + b^2} \)
- \( \mathcal{L}\{e^{at} \sin bt\} = \frac{b}{(s-a)^2 + b^2} \)
- \( \mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s) \)
- \( \mathcal{L}\{e^{at} f(t)\} = F(s-a) \)
- \( \mathcal{L}\{\delta(t-c)\} = e^{-cs} \)
- \( \mathcal{L}\{tf'(t)\} = -F(s) - s \frac{dF(s)}{ds} \)
- \( \mathcal{L}\{f(t-c) \text{ step}(t-c)\} = e^{-cs} F(s) \)
- \( \mathcal{L}\{f(t + c)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\{f'(t)\} + \mathcal{L}\{f''(t)\} + \ldots + \mathcal{L}\{f^{(n)}(t)\} \)
- \( \mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - \ldots - f^{(n-1)}(0) \)