1. [2360/072321 Exam (14 pts)] Let $L(y)$ represent a linear operator describing a fourth order differential equation. Consider the set of solutions to $L(y) = 0$ given by $\{t-1, 2t+1, t^2-7t+3, 4t^2+8t\}$. Does this set constitute a basis for the solution space of $L(y) = 0$? Justify your answer completely.

2. [2360/072321 Exam (20 pts)] A critically damped, unforced harmonic oscillator consisting of a one-quarter kilogram mass and a spring with a restoring constant of 25 newtons per meter is oriented horizontally. Let $x(t)$ be the position of the mass at time t.
   (a) (4 pts) Write the differential equation governing the motion of the oscillator.
   (b) (4 pts) Suppose the motion is started when $t = 0$ by pushing the mass to the left at 10 meters per second from a position 7 meters to the right of the rest position. What are the initial conditions?
   (c) (2 pts) If the oscillator were undamped and driven by the function $f(t) = \cos\omega_f t$, what value of ω_f would result in unbounded solutions to the initial value problem?
   (d) (10 pts) Now suppose an external driving force, $f(t)$, is applied to the oscillator as follows: There is no driving force for the first 5 seconds. At 5 seconds, a driving force of $t-5$ is applied. Five seconds later the driving force is $5e^{-(t-10)}$. Finally, at 20 seconds, a unit impulse is applied. Write this driving force as a single function (not piecewise).

3. [2360/072321 Exam (40 pts)] Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}$.
   (a) (10 pts) One of the eigenvalues of $\mathbf{A}$ is $\lambda = 1$. Calculate $\mathbf{A}\vec{\mathbf{x}}$ where $\vec{\mathbf{x}} = \begin{bmatrix} -1 & 4 & 1 \end{bmatrix}^{\mathrm{T}}$. What can you say about $\vec{\mathbf{x}}$?
   (b) (10 pts) Another eigenvalue of $\mathbf{A}$ is $\lambda = 3$. Suppose after some elementary row operations on the augmented matrix associated with the linear system $(\mathbf{A} - 3\mathbf{I})\,\vec{\mathbf{v}} = \vec{\mathbf{0}}$ you obtain

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 5 & 0 & -5 & 0 \\ 2 & 1 & -4 & 0 \end{array}\right]$$

      i. (5 pts) Put this matrix into RREF.
      ii. (5 pts) Find a basis for the eigenspace associated with $\lambda = 3$. What is its dimension?
   (c) (10 pts) The third eigenvalue of $\mathbf{A}$ is $\lambda = -2$ with associated eigenvector $\vec{\mathbf{v}} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$. Write the general solution of the system of differential equations $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}}$.
   (d) (10 pts) Do the columns of $\mathbf{A}$ form a basis for $\mathbb{R}^3$? Justify your answer.

4. [2360/072321 Exam (30 pts)] Use Laplace transforms to solve the initial value problem

$$y'' + 25y = 25[1 - \text{step}(t-4)],\ y(0) = y'(0) = 0$$

5. [2360/072321 Exam (20 pts)] Solve the initial value problem $\vec{\mathbf{x}}' = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\vec{\mathbf{x}},\ \vec{\mathbf{x}}(0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ knowing that one eigenvalue/eigenvector pair is $\lambda = 1 + i$, $\vec{\mathbf{v}} = \begin{bmatrix} i \\ 1 \end{bmatrix}$.

6. [2360/072321 Exam (26 pts)] Make a legible table on your paper corresponding to the questions below and write the word **TRUE** or **FALSE** in the appropriate place in your table. No partial credit will be awarded and no work is required to be shown.
   (a) Consider the linear system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ where $\mathbf{A}$ is an $n \times n$ matrix.
      i. If $|\mathbf{A}| = 0$, and $\vec{\mathbf{b}} \neq \vec{\mathbf{0}}$ the linear system always has infinitely many solutions.
      ii. If $\mathbf{A}$ has zero as an eigenvalue, the linear system with $\vec{\mathbf{b}} = \vec{\mathbf{0}}$ is consistent.
      iii. The solution to the linear system is always $\vec{\mathbf{x}} = \mathbf{A}^{-1}\vec{\mathbf{b}}$.
      iv. The solution space to the system where $\vec{\mathbf{b}}$ is an $n \times 1$ column vector of ones is a subspace of $\mathbb{R}^n$.
   (b) Consider the system of differential equations $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}}$ where $\mathbf{A}$ is a 2×2 matrix.
      i. If Tr$\mathbf{A} = 0$ and $|\mathbf{A}| \neq 0$, the fixed point at $(0, 0)$ is a stable node.
      ii. If $(\text{Tr}\mathbf{A})^2 - 4|\mathbf{A}| < 0$ and Tr$\mathbf{A} < 0$, then all solutions of the system will approach 0 as $t \to \infty$.
      iii. If $|\mathbf{A}| \neq 0$, the fixed point at $(0, 0)$ in the phase plane is always stable.
      iv. The v and h nullclines are lines.

v. If $\text{Tr}\mathbf{A} = 10$ and $|\mathbf{A}| = 25$ the equilibrium solution is an unstable degenerate node.

(c) Consider the differential equation $y' = 3t^2y^2$.

i. Picard's Theorem guarantees the existence of a unique solution for any initial condition $y(t_0) = y_0$.

ii. The unique solution to the initial value problem consisting of the differential equation and the initial condition $y(0) = 1$ is $y = -(t^3 - 1)^{-1}$.

iii. The equation is a second order linear homogeneous equation.

iv. Euler's method cannot be used to approximate solutions to the differential equation that pass through the origin.

Short table of Laplace Transforms: $\mathscr{L}\{f(t)\} = F(s) \equiv \int_0^\infty e^{-st} f(t)\,\mathrm{d}t$

In this table, a, b, c are real numbers with $c \geq 0$, and $n = 0, 1, 2, 3, \ldots$

$$\mathscr{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}} \qquad \mathscr{L}\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2} \qquad \mathscr{L}\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2}$$

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{\mathrm{d}^n F(s)}{\mathrm{d}s^n} \qquad \mathscr{L}\{e^{at} f(t)\} = F(s-a) \qquad \mathscr{L}\{\delta(t-c)\} = e^{-cs}$$

$$\mathscr{L}\{t f'(t)\} = -F(s) - s\frac{\mathrm{d}F(s)}{\mathrm{d}s} \qquad \mathscr{L}\{f(t-c)\,\text{step}(t-c)\} = e^{-cs}F(s) \qquad \mathscr{L}\{f(t)\,\text{step}(t-c)\} = e^{-cs}\mathscr{L}\{f(t+c)\}$$

$$\mathscr{L}\left\{f^{(n)}(t)\right\} = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \cdots - f^{(n-1)}(0)$$
