1. [2360/070921 Exam (20 pts)] Consider the differential equation $y'' + \frac{y'}{t} = \frac{1}{t^2 \ln t}$, $t > 0$.

   (a) (8 pts) Prove that the set $\{1, \ln t\}$ forms a basis for the solution space of the associated homogeneous equation on the given interval. Fully justify your answer.

   (b) (12 pts) Find the general solution of the differential equation. u-substitution may prove beneficial.

SOLUTION:

(a) Begin by showing that both functions are solutions to the differential equation:

$$(1)'' + \frac{(1)'}{t} = 0 + 0 = 0$$
$$(\ln t)'' + \frac{(\ln t)'}{t} = \left(-\frac{1}{t^2}\right) + \frac{1}{t^2} = 0$$

Check for linear independence.

$$W[1, \ln t](t) = \begin{vmatrix} 1 & \ln t \\ 0 & \frac{1}{t} \end{vmatrix} = \frac{1}{t} \neq 0 \quad \text{for } t > 0$$

implying that the functions do form a basis for the solution space of the associated homogeneous equation since there are two linearly independent functions and the differential equation is order two.

(b) We must use variation of parameters with $y_p = v_1 y_1 + v_2 y_2$ to solve the nonhomogeneous equation. With $y_1 = 1, y_2 = \ln t$ and $f(t) = \frac{1}{t^2 \ln t}$ we have

$$v_1 = \int \frac{-y_2 f}{W[1, \ln t]}\, dt = -\int \frac{\ln t (1/t^2 \ln t)}{1/t}\, dt = -\int \frac{1}{t}\, dt = -\ln|t| = -\ln t \qquad \text{since } t > 0$$
$$v_2 = \int \frac{y_1 f}{W[1, \ln t]}\, dt = \int \frac{1(1/t^2 \ln t)}{1/t}\, dt = \int \frac{1}{t \ln t}\, dt \overset{u=\ln t}{=} \ln|\ln t|$$

Thus

$$y = y_h + y_p = c_1 + \tilde{c}_2 \ln t - \ln t + \ln t\,(\ln|\ln t|) = c_1 + c_2 \ln t + \ln t\,(\ln|\ln t|)$$

where the last equality follows from the fact that we can combine the $\tilde{c}_2$ and -1 into a single new constant multiplying $\ln t$. ■

2. [2360/070921 Exam (38 pts)] An harmonic oscillator is governed by the initial value problem

$$2\ddot{x} + 6\dot{x} + 4x = 80\cos 2t,\ x(0) = 1,\ \dot{x}(0) = 1$$

   (a) (20 pts) Find the position of the mass when $t = \pi$?

   (b) (4 pts) What is the velocity of the mass when $t = \pi$?

   (c) (4 pts) Identify the transient and steady state solutions, if they exist.

   (d) (5 pts) If the oscillator is unforced, but nothing else changes, find the time(s), if any, when the graph of the solution crosses the t-axis.

   (e) (5 pts) If the mass and the forcing function remain the same, write the differential equation that would describe the oscillator in resonance. You need not supply any initial conditions nor solve the resulting differential equation.

SOLUTION:

(a) The characteristic equation of the associated homogeneous equation is $2r^2 + 6r + 4 = 2(r^2 + 3r + 2) = 2(r+1)(r+2) = 0$ with roots $r = -1, -2$ so that $x_h = c_1 e^{-t} + c_2 e^{-2t}$.

The particular solution has the form $x_p = A\cos 2t + B\sin 2t$. Substituting this into the differential equation gives

$$(-4A + 12B)\cos 2t + (-12A - 4B)\sin 2t = 80\cos 2t$$

Equating coefficients on either side of this equation yields the linear system

$$-4A + 12B = 80$$
$$-12A - 4B = 0$$

whose solution is $A = -2, B = 6$. Thus $x_p = -2\cos 2t + 6\sin 2t$ and $x = x_h + x_p = c_1 e^{-t} + c_2 e^{-2t} - 2\cos 2t + 6\sin 2t$.

Now apply the initial conditions.

$$x(0) = c_1 + c_2 - 2 = 1 \implies c_1 + c_2 = 3$$
$$\dot{x}(0) = -c_1 - 2c_2 + 12 = 1 \implies -c_1 - 2c_2 = -11$$

giving $c_1 = -5$ and $c_2 = 8$ so that the displacement is $x(t) = 8e^{-2t} - 5e^{-t} - 2\cos 2t + 6\sin 2t$. The position of the mass at $t = \pi$ is

$$x(\pi) = 8e^{-2\pi} - 5e^{-\pi} - 2$$

(b) The velocity at time t is $\dot{x}(t) = -16e^{-2t} + 5e^{-t} + 4\sin 2t + 12\cos 2t$ so at $t = \pi$ we have

$$\dot{x}(\pi) = 5e^{-\pi} - 16e^{-2\pi} + 12$$

(c)

$$x_{\text{transient}} = 8e^{-2t} - 5e^{-t}$$
$$x_{\text{steady state}} = 6\sin 2t - 2\cos 2t$$

(d) The general solution in this case is $x_h = c_1 e^{-t} + c_2 e^{-2t}$. Applying the initial conditions yields $c_1 = 3$ and $c_2 = -2$. Thus $x(t) = 3e^{-t} - 2e^{-2t}$. The solution will cross the t-axis if/when this vanishes.

$$3e^{-t} - 2e^{-2t} = 0$$
$$e^{-2t}\left(3e^t - 2\right) = 0$$
$$e^t = \frac{2}{3}$$
$$t = \ln\frac{2}{3}$$

As an aside, if we assume only values of $t \geq 0$, the solution does not cross the t-axis and the mass in the oscillator does not pass through its rest position.

(e) To be in resonance, the oscillator must be undamped ($b = 0$) and the circular frequency ($\omega_0 = \sqrt{k/2}$) must be the same as the frequency of the forcing function ($\omega_f = 2$) implying that $k = 8$. The differential equation is then $2\ddot{x} + 8x = 80\cos 2t$.

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3. [2360/070921 Exam (20 pts)] Parts (a), (b), and (c) are not related.

(a) (5 pts) Find a basis for the solution space of the differential equation

$$y^{(5)} - 4y^{(4)} + 13y''' = 0$$

(b) (5 pts) Find a fourth order, constant coefficient, homogeneous linear differential equation whose general solution is

$$y(t) = c_1 e^{2t} + c_2 te^{2t} + c_3\cos t + c_4\sin t$$

(c) (10 pts) Consider the differential equation $y'' - y' - 6y = f(t)$. For each of the following functions, write the form of the particular solution that would be used for solving the nonhomogeneous equation using the method of undetermined coefficients. **DO NOT** solve for the coefficients.

i. $f(t) = t^2 - 1$
ii. $f(t) = 7e^{-2t}$
iii. $f(t) = t^3 e^{3t}$
iv. $f(t) = e^{-4t} - 9$
v. $f(t) = 4\cos 2t + 2e^{-2t}\sin 4t$

SOLUTION:

(a) The characteristic equation is $r^5 - 4r^4 + 13r^3 = r^3(r^2 - 4r + 13) = 0$ with the roots $2 \pm 3i$ each with multiplicity 1, and 0 with multiplicity 3. A basis is $\left\{1, t, t^2, e^{2t}\cos 3t, e^{2t}\sin 3t\right\}$

(b) From the general solution, we can conclude that the roots of the characteristic equation are 2 with multiplicity 2 and $\pm i$. This gives a characteristic equation of $(r-2)^2(r^2+1) = r^4 - 4r^3 + 5r^2 - 4r + 4 = 0$. From this then we have the differential equation

$$y^{(4)} - 4y''' + 5y'' - 4y' + 4y = 0$$

(c) The characteristic equation of the associated homogeneous equation is $r^2 - r - 6 = (r-3)(r+2) = 0$ with roots $r = -2, 3$ giving a basis for the solution space of $\{e^{-2t}, e^{3t}\}$

i. $y_p = At^2 + Bt + C$

ii. $y_p = Ate^{-2t}$

iii. $y_p = t(At^3 + Bt^2 + Ct + D)e^{3t} = (At^4 + Bt^3 + Ct^2 + Dt)e^{3t}$

iv. $y_p = Ae^{-4t} + B$

v. $y_p = A\cos 2t + B\sin 2t + Ce^{-2t}\cos 4t + De^{-2t}\sin 4t$

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4. [2360/070921 Exam (22 pts)] Parts (a) and (b) are not related.

(a) (16 pts) A certain unforced harmonic oscillator is oriented horizontally on a table. It consists of a bucket weighing 1 kilogram attached to a spring, with the spring attached to a wall. Inside the bucket are 6 stones, each having a mass of 1 kilogram. A force of 16 Newtons is required to compress the spring 2 meters and the bucket slides along the table, which offers a damping force equal to 8 times the instantaneous velocity. To start the motion, the bucket is pushed to the left at 2 m/sec from the equilibrium position.

i. (8 pts) Write the governing initial value problem.

ii. (3 pts) Is the oscillator under-, over- or critically damped?

iii. (3 pts) With all else being the same, how many stones must be removed from the bucket to guarantee that the bucket will pass through its rest position at most once?

iv. (2 pts) Is the system conservative? Explain very briefly why or why not.

(b) (6 pts) Convert the initial value problem $y^{(4)} + 2y' - 5y = 0$, $y(1) = 1$, $y'(1) = -2$, $y''(1) = 3$, $y'''(1) = -4$ into a system of first order differential equations with appropriate initial condition(s). Write your answer in terms of matrices, if possible. If not possible, explain why not.

SOLUTION:

(a) i. $k = F/x = 16/2 = 8, m = 1 + 6 = 7, b = 8$. With $x(t)$ representing the displacement of the bucket from its equilibrium position we have

$$7\ddot{x} + 8\dot{x} + 8x = 0 \quad x(0) = 0,\ \dot{x}(0) = -2$$

ii. $b^2 - 4mk = 8^2 - 4(7)(8) = 160 < 0$ so the system is underdamped.

iii. To pass through the rest position at most once, the system needs to be critically damped or overdamped, meaning that $b^2 - 4mk = 64 - 4(m)8 \geq 0$. This will happen if $m \leq 2$. In the context of the problem, we need to remove at least 5 stones from the bucket. This would result in the oscillator being critically damped. If we removed all of the stones the system would be overdamped. In either case, the mass would pass through its rest position at most once.

(b) No. It is damped so it is nonconservative.

(c) Let $u_1 = y, u_2 = y', u_3 = y'', u_4 = y'''$. Then

$$\begin{aligned} u_1' &= y' = u_2 \\ u_2' &= y'' = u_3 \\ u_3' &= y''' = u_4 \\ u_4' &= y^{(4)} = -2y' + 5y = -2u_2 + 5u_1 \end{aligned}$$

with $u_1(1) = 1, u_2(1) = -2, u_3(1) = 3, u_4(1) = -4$. Written as matrices we have

$$\begin{bmatrix} u_1' \\ u_2' \\ u_3' \\ u_4' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 5 & -2 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}, \quad \begin{bmatrix} u_1(1) \\ u_2(1) \\ u_3(1) \\ u_4(1) \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 3 \\ -4 \end{bmatrix}$$

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