1. [2360/070921 Exam (20 pts)] Consider the differential equation $y'' + \dfrac{y'}{t} = \dfrac{1}{t^2 \ln t}$, $t > 0$.

   (a) (8 pts) Prove that the set $\{1, \ln t\}$ forms a basis for the solution space of the associated homogeneous equation on the given interval. Fully justify your answer.

   (b) (12 pts) Find the general solution of the differential equation. u-substitution may prove beneficial.

2. [2360/070921 Exam (38 pts)] An harmonic oscillator is governed by the initial value problem

$$2\ddot{x} + 6\dot{x} + 4x = 80\cos 2t,\ x(0) = 1,\ \dot{x}(0) = 1$$

   (a) (20 pts) Find the position of the mass when $t = \pi$?

   (b) (4 pts) What is the velocity of the mass when $t = \pi$?

   (c) (4 pts) Identify the transient and steady state solutions, if they exist.

   (d) (5 pts) If the oscillator is unforced, but nothing else changes, find the time(s), if any, when the graph of the solution crosses the t-axis.

   (e) (5 pts) If the mass and the forcing function remain the same, write the differential equation that would describe the oscillator in resonance. You need not supply any initial conditions nor solve the resulting differential equation.

3. [2360/070921 Exam (20 pts)] Parts (a), (b), and (c) are not related.

   (a) (5 pts) Find a basis for the solution space of the differential equation

$$y^{(5)} - 4y^{(4)} + 13y''' = 0$$

   (b) (5 pts) Find a fourth order, constant coefficient, homogeneous linear differential equation whose general solution is

$$y(t) = c_1 e^{2t} + c_2 te^{2t} + c_3 \cos t + c_4 \sin t$$

   (c) (10 pts) Consider the differential equation $y'' - y' - 6y = f(t)$. For each of the following functions, write the form of the particular solution that would be used for solving the nonhomogeneous equation using the method of undetermined coefficients. **DO NOT** solve for the coefficients.

      i. $f(t) = t^2 - 1$
      ii. $f(t) = 7e^{-2t}$
      iii. $f(t) = t^3 e^{3t}$
      iv. $f(t) = e^{-4t} - 9$
      v. $f(t) = 4\cos 2t + 2e^{-2t}\sin 4t$

4. [2360/070921 Exam (22 pts)] Parts (a) and (b) are not related.

   (a) (16 pts) A certain unforced harmonic oscillator is oriented horizontally on a table. It consists of a bucket weighing 1 kilogram attached to a spring, with the spring attached to a wall. Inside the bucket are 6 stones, each having a mass of 1 kilogram. A force of 16 Newtons is required to compress the spring 2 meters and the bucket slides along the table, which offers a damping force equal to 8 times the instantaneous velocity. To start the motion, the bucket is pushed to the left at 2 m/sec from the equilibrium position.

      i. (8 pts) Write the governing initial value problem.
      ii. (3 pts) Is the oscillator under-, over- or critically damped?
      iii. (3 pts) With all else being the same, how many stones must be removed from the bucket to guarantee that the bucket will pass through its rest position at most once?
      iv. (2 pts) Is the system conservative? Explain very briefly why or why not.

   (b) (6 pts) Convert the initial value problem $y^{(4)} + 2y' - 5y = 0$, $y(1) = 1$, $y'(1) = -2$, $y''(1) = 3$, $y'''(1) = -4$ into a system of first order differential equations with appropriate initial condition(s). Write your answer in terms of matrices, if possible. If not possible, explain why not.