0. At the top of the first page that you will be scanning and uploading to Gradescope, write the following statement and sign your name to it: "I will abide by the CU Boulder Honor Code on this exam." **FAILURE TO INCLUDE THIS STATEMENT AND YOUR SIGNATURE MAY RESULT IN A PENALTY.**

1. [2360/062521 Exam (18 pts)] Parts (a) and (b) are not related.

   (a) (10 pts) Let $\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 1 & 1 & 0 \end{bmatrix}$

      i. Find all of the eigenvalues of $\mathbf{A}$ and state the multiplicity of each.

      ii. Find a basis for and the dimension of the eigenspace corresponding to the eigenvalue with multiplicity greater than 1.

   (b) (8 pts) Determine if the set of vectors $\left\{1, 1-t, 2-4t+t^2, 6-18t+9t^2-t^3\right\}$ forms a basis for $\mathbb{P}_3$. Be sure to provide correct justification.

2. [2360/062521 Exam (25 pts)] Parts (a) and (b) are not related.

   (a) (15 pts) Consider the vectors $\left\{ \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} \right\}$ in $\mathbb{R}^3$.

      i. (5 pts) Do the vectors form a basis for $\mathbb{R}^3$. Why or why not?

      ii. (5 pts) Find all solutions of $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{0}}$ where the columns of $\mathbf{A}$ are the vectors in the set in the order shown.

      iii. (5 pts) Based on your answer in (ii), find the dimension of and a basis for the subspace consisting of all solutions of the equation $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{0}}$

   (b) (10 pts) Consider the linear system

$$\begin{aligned} x_1 \quad\quad + 3x_3 \quad\quad &= 1 \\ 2x_1 + x_2 + 4x_3 - x_4 &= 2 \\ 3x_1 + 2x_2 + 4x_3 \quad\quad &= -1 \\ 3x_2 - x_3 \quad\quad &= -2 \end{aligned}$$

   Use Cramer's Rule to find $x_2$.

CONTINUED ON NEXT PAGE

3. [2360/062521 Exam (20 pts)] Parts (a), (b), and (c) are not related.

(a) (5 pts) Are the functions $\{1, \sin^2 t, \cos^2 t\}$ linearly independent or linearly dependent on $\mathbb{R}$? Justify your answer.

(b) (10 pts) Given that $\mathbb{M}_{22}$ is the vector space of all $2 \times 2$ matrices, determine if the following subsets, $\mathbb{W}$, are subspaces of $\mathbb{M}_{22}$. Justify your answer.

i. (5 pts) $\mathbb{W}$ is the set of matrices of the form $\begin{bmatrix} a & -b \\ b & c \end{bmatrix}$ where $a, b, c$ are real numbers.

ii. (5 pts) $\mathbb{W}$ is the set of matrices of the form $\begin{bmatrix} 2 & a \\ -a & 3 \end{bmatrix}$ where $a$ is a real number.

(c) (5 pts) Is the set of solutions to the differential equation $y' + (\sin t)y = \cos t$ a vector space? Justify your answer.

4. [2360/062521 Exam (37 pts)] Parts (a), (b) and (c) are not related.

(a) (18 pts) If $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$, calculate the following, if possible. If not possible, simply write "not possible". Hint: for part vi, consider using properties of determinants.

i. $\mathbf{AB}$ ii. $\mathbf{B}^{\mathrm{T}}\mathbf{A}$ iii. $\mathbf{A}\mathbf{A}^{-1}$ iv. $\left|\mathbf{A}^{\mathrm{T}}\mathbf{A}\right|$ v. $(\mathbf{BA})^{\mathrm{T}}$ vi. $\left|\mathbf{B}\mathbf{B}^{\mathrm{T}}\mathbf{B}^{-1}\right|$

(b) (9 pts) The augmented matrix of the linear system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has been transformed, through a number of elementary row operations, to the following:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & k & 1 & 2 \\ 0 & 0 & k-1 & k^2-1 \end{array}\right]$$

where $k$ is a parameter. For which value(s) of $k$, if any, does the system have ...

i. no solution?

ii. exactly one solution?

iii. infinitely many solutions?

(c) (10 pts) You are given the matrices $\mathbf{C}, \mathbf{D}$ and $\mathbf{u}$ as follows:

$$\mathbf{C} = \begin{bmatrix} 2 & 1 & 2 \\ 4 & 2 & 3 \\ 0 & -1 & 1 \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} -\frac{5}{2} & \frac{3}{2} & \frac{1}{2} \\ 2 & -1 & -1 \\ 2 & -1 & 0 \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

i. (4 pts) Compute $\mathbf{DC}$. Be sure to check your answer carefully.

ii. (6 pts) Without performing any elementary row operations or Gauss-Jordan Elimination, and applying what you found in part i, find the solution of $\mathbf{C}\vec{\mathbf{x}} = \vec{\mathbf{u}}$.