1. [APPM 2360 Exam (40 pts)] Consider the differential equation $t^2y'' - 3ty' + 4y = 0, t > 0$.

   (a) Let $\mathbb{S}$ represent the solution space of the differential equation. Is $\mathbb{S}$ a subspace of the vector space $C^2((0,\infty))$, the set of all functions on the interval $(0,\infty)$ having two continuous derivatives? Justify your answer.

   (b) Is the set $\{t^2, t^2\ln t\}$ a basis for $\mathbb{S}$? Justify your answer.

   (c) Find the solution to the differential equation that passes through the point (e, e^2) and whose slope at e is e. To obtain full credit, use Cramer's Rule to aid in finding the solution.

   (d) Now find a particular solution to the equation $t^2y'' - 3ty' + 4y = t^2\ln t, t > 0$. Note that this is a nonhomogeneous version of the original differential equation and you have done most of the work for this already!

SOLUTION:

(a) Let y_1 and y_2 be in $\mathbb{S}$ and let a and b be real numbers. Then

$$\begin{aligned}
&t^2\left(ay_1 + by_2\right)'' - 3t\left(ay_1 + by_2\right)' + 4\left(ay_1 + by_2\right)\\
&= t^2\left(ay_1'' + by_2''\right) - 3t(ay_1' + by_2') + 4(ay_1 + by_2)\\
&= a\left(t^2y_1'' - 3ty_1' + 4y_1\right) + b\left(t^2y_2'' - 3ty_2' + 4y_2\right)\\
&= a(0) + b(0) = 0 \implies ay_1 + by_2 \in \mathbb{S}
\end{aligned}$$

showing that $\mathbb{S}$ is a subspace of $C^2((0,\infty)$.

(b) We need to show that both functions are solutions to the differential equation and that they are linearly independent.

$$t^2(t^2)'' - 3t(t^2)' + 4t^2 = t^2(2) - 3t(2t) + 4t^2 = 0$$

$$t^2(t^2\ln t)'' - 3t(t^2\ln t)' + 4t^2\ln t = t^2(3 + 2\ln t) - 3t(t + 2t\ln t) + 4t^2\ln t = 0$$

$$W[t^2, t^2\ln t](t) = \begin{vmatrix} t^2 & t^2\ln t \\ 2t & t + 2t\ln t \end{vmatrix} = t^3$$

Since the Wronskian is nonzero for $t > 0$ the functions are linearly independent. Since both of them solve the differential equation and the differential equation is second order, meaning that $\mathbb{S}$ has dimension 2, $\{t^2, t^2\ln t\}$ is a basis for $\mathbb{S}$.

(c) The general solution is $y(t) = c_1t^2 + c_2t^2\ln t$. From this we have $y'(t) = 2c_1t + c_2(t + 2t\ln t)$. Applying the initial conditions leads to the system of equations

$$\begin{aligned} c_1e^2 + c_2e^2 &= e^2 \\ 2c_1e + c_2(e + 2e) &= e \end{aligned} \implies \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Cramer's Rule then gives

$$c_1 = \frac{\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}} = 2 \qquad c_2 = \frac{\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}} = -1$$

so that the solution to the initial value problem is $y(t) = 2t^2 - t^2\ln t = t^2(2 - \ln t)$

(d) Since this is variable coefficient problem, we must use variation of parameters. Noting that the ODE as given does not have a coefficient of 1 on the second derivative term, we divide by t^2, giving $f(t) = \ln t$ as the nonhomogeneous (forcing) term. With $y_1 = t^2$ and $y_2 = t^2\ln t$ as the two solutions to the associated homogeneous equation we have

$$v_1' = -\frac{y_2f}{W} = -\frac{(t^2\ln t)(\ln t)}{t^3} = -\frac{(\ln t)^2}{t} \implies v_1 = -\int\frac{(\ln t)^2}{t}\mathrm{d}t \overset{u=\ln t}{=} -\frac{1}{3}(\ln t)^3$$

$$v_2' = \frac{y_1f}{W} = \frac{(t^2)(\ln t)}{t^3} = \frac{\ln t}{t} \implies v_2 = \int\frac{\ln t}{t}\,\mathrm{d}t \overset{u=\ln t}{=} (\ln t)^2$$

so that

$$y_p = v_1y_1 + v_2y_2 = -\frac{1}{3}(\ln t)^3(t^2) + (\ln t)^2\left(t^2\ln t\right) = \frac{2}{3}t^2(\ln t)^3$$

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2. [APPM 2360 Exam (20 pts)] Consider the matrix $\mathbf{A} = \begin{bmatrix} k & 0 & 2 \\ 0 & k & 3 \\ -1 & k & 4 \end{bmatrix}$.

(a) For what value(s) of k does $\mathbf{A}^{-1}$ exist?

(b) For what value(s) of k does the linear system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{0}}$ have nontrivial solutions?

(c) For what value(s) of k does $\mathbf{A}$ have zero as an eigenvalue?

(d) For any vector $\vec{\mathbf{b}} \in \mathbb{R}^3$, find the value(s) of k for which the linear system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has a unique solution.

SOLUTION:

Compute the determinant by expanding along the first column:

$$|\mathbf{A}| = (-1)^{1+1}k \begin{vmatrix} k & 3 \\ k & 4 \end{vmatrix} + (-1)^{3+1}(-1) \begin{vmatrix} 0 & 2 \\ k & 3 \end{vmatrix} = k^2 + 2k = k(k+2)$$

(a) Need $|\mathbf{A}| \neq 0$. Thus, $k \neq 0, -2$.

(b) Need $|\mathbf{A}| = 0$. Thus, $k = 0, -2$.

(c) Need $|\mathbf{A}| = 0$. Thus, $k = 0, -2$.

(d) Need $|\mathbf{A}| \neq 0$. Thus, $k \neq 0, -2$.

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3. [APPM 2360 Exam (30 pts)] The following problems are not related.

(a) Suppose that a spring requires 32 N to stretch it 32 m. A 2 kg mass is attached to the spring and the entire apparatus is immersed in a medium that offers a damping force proportional to 10 times the instantaneous velocity. The mass is initially released from a point 2 m below the equilibrium position with a downward velocity of 3 m/s. (Use the convention that displacements measured below the equilibrium are positive). During the first $\pi/2$ seconds an external (driving) force $f(t) = 16t$ is applied to the mass-spring system. After $\pi/2$ seconds the external force is replaced by the constant force $f(t) = 8\pi$. Set up, **but do not solve**, the initial value problem that models this situation. Use an appropriate step function to describe the external forcing.

(b) A 1000 gallon tank initially contains 50 pounds of salt dissolved in 200 gallons of water. Brine containing 2 pounds of salt per gallon enters the tank at the rate of 20 gallons per hour, beginning when $t = 0$. The well-mixed brine leaves the tank at 10 gallons per hour.

i. Find the volume $V(t)$ of brine in the tank at time t.

ii. Set up, **but do not solve**, the initial value problem that determines the amount of salt, $x(t)$, in the tank at any time t.

iii. Over what time interval is the differential equation in part (b) valid?

SOLUTION:

(a) To find the spring constant, $32 = k(32) \implies k = 1$. Let $y(t)$ be the displacement of the mass.

$$\begin{aligned} 2y'' + 10y' + y &= 16t - 16t\,\text{step}(t - \pi/2) + 8\pi\,\text{step}(t - \pi/2), \qquad y(0) = 2, y'(0) = 3 \\ &= 16\left[t - (t - \pi/2)\text{step}(t - \pi/2)\right] \end{aligned}$$

(b) i. 20 gallons per hour going into the tank, 10 gallons per hour going out. Thus the change in volume of the brine in the tank is $20 - 10 = 10$ gallons per hour. The tank starts with 200 gallons in it. Thus $V(t) = 200 + 10t$.

ii. Rate of change equals rate in minus rate out.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \left(2\frac{\text{lb}}{\text{gal}}\right)\left(20\frac{\text{gal}}{\text{hour}}\right) - \left(\frac{x(t)}{200 + 10t}\frac{\text{lb}}{\text{gal}}\right)\left(10\frac{\text{gal}}{\text{hour}}\right)$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{x}{20 + t} = 40, \quad x(0) = 50$$

iii. The equation is valid until the tank is filled, which occurs when $1000 = 200 + 10t \implies t = 80$. The valid interval is thus $0 \leq t \leq 80$.

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4. [APPM 2360 (30 pts)] The following problems are not related.

(a) Solve the initial value problem $\ddot{x} + 2\dot{x} + x = \delta(t - 1),\ x(0) = 2, \dot{x}(0) = 3$.

(b) The time rate of change of a certain population is proportional to the product of the population at time t and $\cos t$, where t is measured in minutes. At the initial time of $t = 0$ the population is 10 individuals. When $t = \pi/2$ minutes the population is $10e$ individuals. What is the population when the time reaches $3\pi/4$ minutes?

SOLUTION:

(a) Use Laplace Transforms.

$$s^2X(s) - sx(0) - \dot{x}(0) + 2\left[sX(s) - x(0)\right] + X(s) = e^{-s}$$
$$\left(s^2 + 2s + 1\right) X(s) = e^{-s} + sx(0) + \dot{x}(0) + 2x(0)$$
$$X(s) = \frac{e^{-s}}{(s+1)^2} + \frac{2s+7}{(s+1)^2} \qquad \text{(partial fractions on second term)}$$
$$X(s) = \frac{e^{-s}}{(s+1)^2} + \frac{2}{s+1} + \frac{5}{(s+1)^2}$$
$$x(t) = \mathscr{L}^{-1}\left\{X(s)\right\} = \mathscr{L}^{-1}\left\{\frac{e^{-s}}{(s+1)^2}\right\} + \mathscr{L}^{-1}\left\{\frac{2}{s+1}\right\} + \mathscr{L}^{-1}\left\{\frac{5}{(s+1)^2}\right\}$$
$$x(t) = (t-1)e^{-(t-1)}\text{step}(t-1) + 2e^{-t} + 5te^{-t}$$

(b) Let $P(t)$ be the number of individuals at time t.

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP\cos t \qquad \text{(separable)}$$
$$\int \frac{\mathrm{d}P}{P} = k\int \cos t \,\mathrm{d}t$$
$$\ln|P| = k\sin t + c$$
$$P(t) = Ce^{k\sin t} \qquad (C = e^c)$$
$$P(0) = 10 = Ce^0 \implies C = 10$$
$$P(t) = 10e^{k\sin t}$$
$$P(\pi/2) = 10e = 10e^{k(1)} \implies k = 1$$
$$P(t) = 10e^{\sin t} \implies P(3\pi/4) = 10e^{\sqrt{2}/2}$$

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5. [APPM 2360 Exam (30 pts)] Consider the initial value problem $\vec{\mathbf{x}}' = \begin{bmatrix} 2 & 5 \\ -1 & 4 \end{bmatrix} \vec{\mathbf{x}}$, $\vec{\mathbf{x}}(0) = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$.

(a) [(8 pts)] Using the trace-determinant plane, classify the geometry and stability of the equilibrium solution/fixed point $(0,0)$.

(b) [(22 pts)] Find $\vec{\mathbf{x}}(t)$, writing your answer as a single vector.

SOLUTION:

(a) Since Tr$\mathbf{A} = 6$, $|\mathbf{A}| = 13$, and $(\text{Tr}\mathbf{A})^2 - 4|\mathbf{A}| = -16 < 0$, the equilibrium solution is a repelling spiral.

(b) The eigenvalues of $\mathbf{A}$ are $3 \pm 2i$. To find the eigenvector associated with $3 + 2i$,

$$\left[\begin{array}{cc|c} 2-(3+2i) & 5 & 0 \\ -1 & 4-(3+2i) & 0 \end{array}\right] = \left[\begin{array}{cc|c} -1-2i & 5 & 0 \\ -1 & 1-2i & 0 \end{array}\right] \to \left[\begin{array}{cc|c} -1 & 1-2i & 0 \\ -1-2i & 5 & 0 \end{array}\right] {\scriptstyle R_2^* = (-1-2i)R_1 + R_2}$$
$$\to \left[\begin{array}{cc|c} -1 & 1-2i & 0 \\ 0 & 0 & 0 \end{array}\right] \implies \vec{\mathbf{v}} = \begin{bmatrix} 1-2i \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + i\begin{bmatrix} -2 \\ 0 \end{bmatrix}$$
$$\vec{\mathbf{x}}(t) = c_1e^{3t}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\cos 2t - \begin{bmatrix} -2 \\ 0 \end{bmatrix}\sin 2t\right) + c_2e^{3t}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\sin 2t + \begin{bmatrix} -2 \\ 0 \end{bmatrix}\cos 2t\right)$$
$$\vec{\mathbf{x}}(0) = \begin{bmatrix} 10 \\ 0 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} -2 \\ 0 \end{bmatrix} \implies \begin{array}{c} c_1 - 2c_2 = 10 \\ c_1 = 0 \end{array} \implies c_1 = 0, c_2 = 5$$
$$\vec{\mathbf{x}}(t) = e^{3t}\begin{bmatrix} 10\cos 2t - 5\sin 2t \\ -5\sin 2t \end{bmatrix}$$

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