1. [APPM 2360 Exam (40 pts)] Consider the differential equation $t^2y'' - 3ty' + 4y = 0, t > 0$.

   (a) Let $\mathbb{S}$ represent the solution space of the differential equation. Is $\mathbb{S}$ a subspace of the vector space $C^2((0, \infty))$, the set of all functions on the interval $(0, \infty)$ having two continuous derivatives? Justify your answer.

   (b) Is the set $\{t^2, t^2 \ln t\}$ a basis for $\mathbb{S}$? Justify your answer.

   (c) Find the solution to the differential equation that passes through the point (e, e^2) and whose slope at e is e. To obtain full credit, use Cramer's Rule to aid in finding the solution.

   (d) Now find a particular solution to the equation $t^2y'' - 3ty' + 4y = t^2 \ln t, t > 0$. Note that this is a nonhomogeneous version of the original differential equation and you have done most of the work for this already!

2. [APPM 2360 Exam (20 pts)] Consider the matrix $\mathbf{A} = \begin{bmatrix} k & 0 & 2 \\ 0 & k & 3 \\ -1 & k & 4 \end{bmatrix}$.

   (a) For what value(s) of k does $\mathbf{A}^{-1}$ exist?

   (b) For what value(s) of k does the linear system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{0}}$ have nontrivial solutions?

   (c) For what value(s) of k does $\mathbf{A}$ have zero as an eigenvalue?

   (d) For any vector $\vec{\mathbf{b}} \in \mathbb{R}^3$, find the value(s) of k for which the linear system $\mathbf{A}\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has a unique solution.

3. [APPM 2360 Exam (30 pts)] The following problems are not related.

   (a) Suppose that a spring requires 32 N to stretch it 32 m. A 2 kg mass is attached to the spring and the entire apparatus is immersed in a medium that offers a damping force proportional to 10 times the instantaneous velocity. The mass is initially released from a point 2 m below the equilibrium position with a downward velocity of 3 m/s. (Use the convention that displacements measured below the equilibrium are positive). During the first $\pi/2$ seconds an external (driving) force $f(t) = 16t$ is applied to the mass-spring system. After $\pi/2$ seconds the external force is replaced by the constant force $f(t) = 8\pi$. Set up, **but do not solve**, the initial value problem that models this situation. Use an appropriate step function to describe the external forcing.

   (b) A 1000 gallon tank initially contains 50 pounds of salt dissolved in 200 gallons of water. Brine containing 2 pounds of salt per gallon enters the tank at the rate of 20 gallons per hour, beginning when $t = 0$. The well-mixed brine leaves the tank at 10 gallons per hour.

      i. Find the volume $V(t)$ of brine in the tank at time t.

      ii. Set up, **but do not solve**, the initial value problem that determines the amount of salt, $x(t)$, in the tank at any time t.

      iii. Over what time interval is the differential equation in part (b) valid?

4. [APPM 2360 (30 pts)] The following problems are not related.

   (a) Solve the initial value problem $\ddot{x} + 2\dot{x} + x = \delta(t-1),\ x(0) = 2, \dot{x}(0) = 3$.

   (b) The time rate of change of a certain population is proportional to the product of the population at time t and $\cos t$, where t is measured in minutes. At the initial time of $t = 0$ the population is 10 individuals. When $t = \pi/2$ minutes the population is $10e$ individuals. What is the population when the time reaches $3\pi/4$ minutes?

5. [APPM 2360 Exam (30 pts)] Consider the initial value problem $\vec{\mathbf{x}}' = \begin{bmatrix} 2 & 5 \\ -1 & 4 \end{bmatrix} \vec{\mathbf{x}},\ \vec{\mathbf{x}}(0) = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$.

   (a) [(8 pts)] Using the trace-determinant plane, classify the geometry and stability of the equilibrium solution/fixed point $(0, 0)$.

   (b) [(22 pts)] Find $\vec{\mathbf{x}}(t)$, writing your answer as a single vector.