1. [APPM 2360 Exam (20 pts)] Consider the nonhomogeneous differential equation $t^2y'' - 3ty' + 3y = t^4,\ t > 0$.

   (a) Assuming solutions of the form $y = t^r$, solve the associated homogeneous equation.

   (b) Show that your solutions from part (a) form a basis for the solution space of the homogeneous equation.

   (c) Find a particular solution to the nonhomogeneous equation.

   (d) Solve the initial value problem consisting of the nonhomogeneous differential equation along with the initial conditions $y(1) = -\frac{2}{3}, y'(1) = \frac{7}{3}$.

SOLUTION:

(a) Letting $y = t^r$ gives the characteristic equation $r^2 - 4r + 3 = (r-3)(r-1) = 0 \implies r = 1, 3$. We thus have $y(t) = t, t^3$ so the solution to the homogeneous equation is $y(t) = c_1 t + c_2 t^3$.

(b) The solution space is dimension 2 and we have two solutions so we need only check that $y_1 = t$ and $y_2 = t^3$ are linearly independent.

$$W[t, t^3](t) = \begin{vmatrix} t & t^3 \\ 1 & 3t^2 \end{vmatrix} = 2t^3$$

Since this is nonzero for $t > 0$, t and t^3 are linearly independent so that $\{t, t^3\}$ forms a basis for the solution space of the homogeneous equation.

(c) Since this is variable coefficient problem, we must use variation of parameters. Noting that the ODE as given does not have a coefficient of 1 on the second derivative term, we divide by t^2 which gives $f(t) = t^2$ as the nonhomogeneous (forcing) term.

$$v_1' = -\frac{y_2 f}{W} = -\frac{(t^3)(t^2)}{2t^3} = -\frac{t^2}{2} \implies v_1 = -\frac{1}{2}\int t^2\,\mathrm{d}t = -\frac{1}{6}t^3$$

$$v_2' = \frac{y_1 f}{W} = \frac{(t)(t^2)}{2t^3} = \frac{1}{2} \implies v_2 = \frac{1}{2}\int \mathrm{d}t = \frac{1}{2}t$$

so that

$$y_p = v_1 y_1 + v_2 y_2 = \left(-\frac{1}{6}t^3\right)t + \left(\frac{1}{2}t\right)t^3 = \frac{1}{3}t^4$$

(d) The general solution to the ODE is $y(t) = c_1 t + c_2 t^3 + \frac{1}{3}t^4$. Applying the initial conditions yields

$$y(1) = -\frac{2}{3} = c_1 + c_2 + \frac{1}{3} \implies c_1 + c_2 = -1$$

$$y'(1) = \frac{7}{3} = c_1 + 3c_2 + \frac{4}{3} \implies c_1 + 3c_2 = 1$$

with solution $c_1 = -2, c_2 = 1$ so that $y(t) = -2t + t^3 + \frac{1}{3}t^4$.

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2. [APPM 2360 Exam (20 pts)] Find the general solution of $\frac{\mathrm{d}^4 y}{\mathrm{d}t^4} - 4\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = 12t - 16$. Use the Method of Undetermined Coefficients to find a particular solution.

SOLUTION:

The characteristic equation associated with the homogeneous differential equation is $r^4 - 4r^2 = r^2(r-2)(r+2) = 0$ with roots $r = -2, 2$, each having multiplicity 1, and $r = 0$ with multiplicity 2. The general solution to the homogeneous equation is thus

$$y_h(t) = c_1 + c_2 t + c_3 e^{2t} + c_4 e^{-2t}$$

We choose $y_p = t^2(At + B) = At^3 + Bt^2$. Substitution into the ODE yields

$$y_p^{(4)} - 4y_p'' = -4(6At + 2B) = 12t - 16 \implies A = -\frac{1}{2}, B = 2 \implies y_p(t) = -\frac{1}{2}t^3 + 2t^2$$

so that the general solution to the nonhomogeneous equation is

$$y(t) = y_h(t) + y_p(t) = c_1 + c_2 t + c_3 e^{2t} + c_4 e^{-2t} - \frac{1}{2}t^3 + 2t^2$$

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3. [APPM 2360 Exam (20 pts)] Consider the linear operator $L(\vec{\mathbf{y}}) = y''' - 6y'' + 13y' - 10y$.

(a) (7 pts) If $y = e^{2t}$ is one solution to the equation $L(\vec{\mathbf{y}}) = 0$, find the general solution of $L(\vec{\mathbf{y}}) = 0$.

(b) (8 pts) Now consider $L(\vec{\mathbf{y}}) = f(t)$. Write down the form of the particular solution y_p to use in the Method of Undetermined Coefficients for the given $f(t)$. Do not find the constants.

i. $f(t) = 5e^t + e^{2t} - 1$.

ii. $f(t) = te^{2t}$

iii. $f(t) = \cos 5t + \sin 7t$

iv. $f(t) = 10e^{2t}\sin t$

(c) (5 pts) Convert the initial value problem $L(\vec{\mathbf{y}}) = e^{-t} + 6,\ y(0) = 4, y'(0) = 3, y''(0) = 7$ into a system of three first order differential equations, writing your answer in the form $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}} + \vec{\mathbf{f}}(t)$. Be sure to include the initial condition in your answer.

SOLUTION:

(a) Since $y(t) = e^{2t}$ is a solution to the homogeneous equation, $r = 2$ is solution to the characteristic equation $r^3 - 6r^2 + 13r - 10 = 0$. Knowing this allows us to write the characteristic equation as $(r-2)(r^2 - 4r + 5) = 0$. Using the quadratic formula on the second factor yields $r = 2 \pm i$ so that the other solutions to the homogeneous equation are $y(t) = e^{2t}\cos t$ and $y(t) = e^{2t}\sin t$. The general solution to $L(\vec{\mathbf{y}}) = 0$, is thus $y(t) = e^{2t}\left(c_1 + c_2\cos t + c_3 \sin t\right)$.

(b) i. $y_p(t) = Ae^t + Bte^{2t} + C$

ii. $y_p(t) = t(At + B)e^{2t}$

iii. $y_p(t) = A\cos 5t + B\sin 5t + C\cos 7t + D\sin 7t$

iv. $y_p(t) = t\left(Ae^{2t}\sin t + Be^{2t}\cos t\right)$

(c) Let $x_1 = y, x_2 = y', x_3 = y''$ and rewrite the differential equation as $y''' = e^{-t} + 6 + 6y'' - 13y' + 10y$. Then

$$\begin{aligned} x_1' &= y' = x_2 \\ x_2' &= y'' = x_3 \\ x_3' &= y''' = e^{-t} + 6 + 6x_3 - 13x_2 + 10x_1 \end{aligned}$$

Writing this as a matrix equation yields

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 10 & -13 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ e^{-t} + 6 \end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix}$$

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4. [APPM 2360 (20 pts)] The following problems are not related.

(a) Consider an harmonic oscillator governed by the differential equation $m\ddot{x} + b\dot{x} + x = A\cos\left(\frac{1}{4}t\right)$.

i. (3 pts) Find the values of A, m and b so that the oscillator will exhibit resonance.

ii. (3 pts) Find the values of A, m and b so that the equation will have bounded solutions.

iii. (2 pts) If the mass of the oscillator is 1 unit, find the values of A and b such that the oscillator will be unforced and the mass will pass through the equilibrium position as most once.

(b) (12 pts) Now consider an undamped oscillator with mass 1 unit, restoring/spring constant 1 unit that starts from rest at the equilibrium position. It is oriented horizontally and is driven by the function $f(t) = \sin t$. After $3\pi/2$ units of time have elapsed:

i. In relation to the equilibrium position, where is the mass?

ii. How fast and in what direction is the mass moving?

SOLUTION:

(a) i. System needs to be undamped, so $b = 0$. System needs to be forced, so $A \neq 0$. Forcing frequency must equal circular frequency, so $\frac{1}{4} = \sqrt{\frac{1}{m}} \implies m = 16$.

ii. If $A = 0$, then $m > 0, b \geq 0$. If $A \neq 0$, then if $b = 0$, $m \neq 16$; if $b \neq 0, m > 0$.

iii. Unforced means $A = 0$. For the mass to pass through the equilibrium position at most once requires the system to be either critically or overdamped so $b^2 - 4mk = b^2 - 4(1)(1) \geq 0 \implies b \geq 2$.

(b) The initial value problem governing this situation is $\ddot{x} + x = \sin t,\ x(0) = \dot{x}(0) = 0$. The solution to the homogeneous equation is $x(t) = c_1\cos t + c_2 \sin t$. Substituting $x_p = At\cos t + Bt\sin t$ into the nonhomogeneous equation yields $A = -\frac{1}{2}, B = 0$. The motion is then described by $x(t) = c_1\cos t + c_2\sin t - \frac{1}{2}t\cos t$. Applying the initial conditions yields $c_1 = 0, c_2 = \frac{1}{2}$. Thus $x(t) = \frac{1}{2}\left(\sin t - t\cos t\right)$ and $\dot{x}(t) = \frac{1}{2}t\sin t$.

i. $x(3\pi/2) = -1/2$ so the mass is $1/2$ unit to the left of the equilibrium position.

ii. $\dot{x}(3\pi/2) = -3\pi/4$ so the mass is moving to the left at $3\pi/4$ units.

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5. [APPM 2360 Exam (20 pts)] A mass of 1 kg is attached to a spring whose constant is 5 N/m. Initially, the mass is released 1 m below the equilibrium position with a downward velocity of 5 m/s, and the subsequent motion takes place in a medium that offers a damping force that is numerically equal to 2 times the instantaneous velocity.

(a) (15 pts) Find the equation of motion if the mass is driven by an external force equal to $f(t) = 12\cos 2t + 3\sin 2t$.

(b) (5 pts) Find the amplitude of the steady-state solution.

SOLUTION:

(a) The initial value problem is $\ddot{x} + 2\dot{x} + 5x = 12\cos 2t + 3\sin 2t,\ x(0) = 1, \dot{x}(0) = 5$. The roots of the characteristic equation are $r = -1 \pm 2i$ giving the solution to the homogeneous equation as $x_h(t) = e^{-t}(c_1 \cos 2t + c_2 \sin 2t)$. Substituting $x_p(t) = A\cos 2t + B\sin 2t$ into the nonhomogeneous equation yields $A = 0, B = 3$ so that $x_p(t) = 3\sin 2t$ and $x(t) = e^{-t}(c_1 \cos 2t + c_2 \sin 2t) + 3\sin 2t$. Application of the initial conditions shows that $c_1 = 1$ and $c_2 = 0$. The equation of motion is thus $x(t) = e^{-t}\cos 2t + 3\sin 2t$.

(b) The steady-state solution is the one that remains after a long time which is $3\sin 2t$, the amplitude of which is 3.

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