1. [APPM 2360 Exam (20 pts)] Let $\vec{\mathbf{u}} = \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \\ 0 \end{bmatrix}$

   (a) (6 pts) Show that $\vec{\mathbf{u}}^{\mathrm{T}}\vec{\mathbf{u}} = \mathbf{I}_1$, where $\mathbf{I}_1$ is the 1×1 identity matrix.

   (b) (6 pts) Compute $\mathbf{H} = \mathbf{I}_3 - 2\vec{\mathbf{u}}\,\vec{\mathbf{u}}^{\mathrm{T}}$

   (c) (6 pts) Show that $\mathbf{H}$ is nonsingular (invertible).

   (d) (2 pts) Can Cramer's Rule be used to solve the system $\left(\vec{\mathbf{u}}\,\vec{\mathbf{u}}^{\mathrm{T}}\right)\vec{\mathbf{x}} = \vec{\mathbf{b}}$? Explain why or why not.

SOLUTION:

(a)

$$\vec{\mathbf{u}}^{\mathrm{T}}\vec{\mathbf{u}} = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \\ 0 \end{bmatrix} = \left[\left(\tfrac{\sqrt{2}}{2}\right)\left(\tfrac{\sqrt{2}}{2}\right) + \left(\tfrac{\sqrt{2}}{2}\right)\left(\tfrac{\sqrt{2}}{2}\right) + (0)(0) \right] = [1]$$

(b)

$$\vec{\mathbf{u}}\,\vec{\mathbf{u}}^{\mathrm{T}} = \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \\ 0 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 & 0 \end{bmatrix} = \begin{bmatrix} \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) & \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) & \left(\frac{\sqrt{2}}{2}\right)(0) \\ \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) & \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) & \left(\frac{\sqrt{2}}{2}\right)(0) \\ (0)\left(\frac{\sqrt{2}}{2}\right) & (0)\left(\frac{\sqrt{2}}{2}\right) & (0)(0) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so that

$$\mathbf{H} = \mathbf{I}_3 - 2\vec{\mathbf{u}}\,\vec{\mathbf{u}}^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(c)

$$|\mathbf{H}| = (-1)(-1)^{2+1}\begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} = -1 \neq 0 \implies \mathbf{H} \text{ is nonsingular}$$

(d) Although $\vec{\mathbf{u}}\,\vec{\mathbf{u}}^{\mathrm{T}}$ is a square matrix, it contains a row of zeros, implying that it is singular. Consequently, Cramer's Rule cannot be used to solve the given system.

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2. [APPM 2360 Exam (15 pts)] Consider the linear system

$$\begin{aligned} x - 4y &= 17 \\ 3x - 12y &= k \\ -2x + 8y &= -34 \end{aligned}$$

   (a) (8 pts) Use Gauss-Jordan Reduction to determine the value of k that makes the system consistent.

   (b) (7 pts) Using the value of k found in part (a), write the solution to the system using the Nonhomogenous Principle $\vec{\mathbf{x}} = \vec{\mathbf{x}}_h + \vec{\mathbf{x}}_p$.

SOLUTION:

(a) Converting the augmented matrix for this system this into RREF yields

$$\left[\begin{array}{rr|r} 1 & -4 & 17 \\ 3 & -12 & k \\ -2 & 8 & -34 \end{array}\right] \overset{R_2^* = -3R_1 + R_2}{\underset{R_3^* = 2R_1 + R_3}{\Rightarrow}} \left[\begin{array}{rr|c} 1 & -4 & 17 \\ 0 & 0 & k - 51 \\ 0 & 0 & 0 \end{array}\right]$$

The system is consistent if $k = 51$.

(b) Using the RREF with $k = 51$, x is the basic variable and y is the free variable. Setting $y = t$, particular solutions are of the form

$$\vec{\mathbf{x}}_p = \begin{bmatrix} 17 + 4t \\ t \end{bmatrix} \text{ and with } t = 0 \text{ we have } \vec{\mathbf{x}}_p = \begin{bmatrix} 17 \\ 0 \end{bmatrix}$$

Again from the RREF, we have

$$\vec{\mathbf{x}}_h = \begin{bmatrix} 4t \\ t \end{bmatrix} = t\begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

so that

$$\vec{\mathbf{x}} = \vec{\mathbf{x}}_h + \vec{\mathbf{x}}_p = t\begin{bmatrix} 4 \\ 1 \end{bmatrix} + \begin{bmatrix} 17 \\ 0 \end{bmatrix}, \quad t \in \mathbb{R}$$

3. [APPM 2360 Exam (20 pts)] Use the matrix inverse to find the solution to the system

$$\begin{aligned} x_1 \quad + 2x_2 + 3x_3 &= 3 \\ x_2 + 2x_3 &= 1 \\ -2x_1 \qquad + 3x_3 &= 4 \end{aligned}$$

SOLUTION:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ -2 & 0 & 3 & 0 & 0 & 1 \end{array}\right] \begin{smallmatrix} R_3^*=2R_1+R_3 \end{smallmatrix} \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 4 & 9 & 2 & 0 & 1 \end{array}\right] \begin{smallmatrix} R_3^*=-4R_2+R_3 \end{smallmatrix} \Rightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & -4 & 1 \end{array}\right] \begin{smallmatrix} R_1^*=-3R_3+R_1 \\ R_2^*=-2R_3+R_2 \end{smallmatrix} \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -5 & 12 & -3 \\ 0 & 1 & 0 & -4 & 9 & -2 \\ 0 & 0 & 1 & 2 & -4 & 1 \end{array}\right] \begin{smallmatrix} R_1^*=-2R_2+R+1 \end{smallmatrix} \Rightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -6 & 1 \\ 0 & 1 & 0 & -4 & 9 & -2 \\ 0 & 0 & 1 & 2 & -4 & 1 \end{array}\right] \Longrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 & -6 & 1 \\ -4 & 9 & -2 \\ 2 & -4 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ -11 \\ 6 \end{bmatrix}$$

4. [APPM 2360 (25 pts)] The following problems are not related.

(a) (12 pts) Suppose $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are $n \times n$ matrices with $|\mathbf{A}| = 3$, $|\mathbf{B}| = 1$, $|\mathbf{C}| = 0$. Calculate the following or explain why they fail to exist.

i. $|\mathbf{AB}|$
ii. $|\mathbf{B}^{\mathrm{T}}|$
iii. $|\mathbf{B}^2\mathbf{A}\mathbf{C}^{-1}|$
iv. $|\mathbf{D}|$, where $\mathbf{D}$ is the matrix obtained by interchanging the second and n^{th} rows of $\mathbf{A}$

(b) (9 pts) Let $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 3 & 2 \\ -1 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 4 & -2 \\ 1 & 1 \\ 0 & 3 \end{bmatrix}$. Calculate the following or explain why they fail to exist.

i. $\mathbf{A}\mathbf{B}^{\mathrm{T}}$
ii. $\mathbf{A}^{\mathrm{T}} + \mathbf{B}$
iii. $\left(\mathbf{B}^{\mathrm{T}}\right)^{-1}$

(c) (4 pts) Let $\mathbf{A}$ be an $n \times n$ invertible matrix satisfying $\mathbf{A}^3 + 2\mathbf{A} = \mathbf{I}$. Find an expression for $\mathbf{A}^{-1}$.

SOLUTION:

(a) i. $|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}| = (3)(1) = 3$
ii. $|\mathbf{B}^{\mathrm{T}}| = |\mathbf{B}| = 1$
iii. $\mathbf{C}$ is not invertible so $|\mathbf{B}^2\mathbf{A}\mathbf{C}^{-1}|$ does not exist.
iv. $|\mathbf{D}| = -|\mathbf{A}| = -3$

(b) i. $\mathbf{A}\mathbf{B}^{\mathrm{T}} = \begin{bmatrix} 0 & 1 \\ 3 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 0 \\ -2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 3 \\ 8 & 5 & 6 \\ -4 & -1 & 0 \end{bmatrix}$
ii. $\mathbf{A}^{\mathrm{T}}$ and $\mathbf{B}$ are not the same order and thus cannot added
iii. $\mathbf{B}$ is not square nor is $\mathbf{B}^{\mathrm{T}}$ so its inverse does not exist

(c) $\mathbf{A}^3 + 2\mathbf{A} = \mathbf{A}\left(\mathbf{A}^2 + 2\mathbf{I}\right) = \left(\mathbf{A}^2 + 2\mathbf{I}\right)\mathbf{A} = \mathbf{I} \Longrightarrow \mathbf{A}^{-1} = \mathbf{A}^2 + 2\mathbf{I}$

5. [APPM 2360 Exam (20 pts)] The following problems are not related.

(a) (8 pts) Decide if the following subsets $\mathbb{W}$ of the given vector space $\mathbb{V}$ are subspaces. Assume that the standard operations of vector addition and scalar multiplication apply. Justify the correct answer completely for full credit. A simple yes/no will result in zero points.

i. $\mathbb{V} = C([0,1])$; $\quad \mathbb{W} = \left\{ f(t) \,\middle|\, \int_0^1 f(t)\,\mathrm{d}t = 2 \right\}$

ii. $\mathbb{V} = \mathbb{M}_{23}$; $\quad \mathbb{W} =$ matrices of the form $\begin{bmatrix} 0 & a & b \\ c & 0 & d \end{bmatrix}$ where a, b, c, d are real numbers.

(b) (4 pts) Determine whether or not the set $S = \{2, 1-t, t+t^3\}$ forms a basis for some vector space. If so, what is its dimension? If not, explain why not.

(c) (8 pts) Find the eigenvalues and eigenvectors of $\mathbf{A} = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$.

SOLUTION:

(a) i. The zero vector, $f(t) \equiv 0$, is not in $\mathbb{W}$ since $\int_0^1 0\,\mathrm{d}t = 0 \neq 2$ so the subset is not a subspace.

ii. The zero vector is clearly an element of $\mathbb{W}$. To verify closure, let $\vec{\mathbf{u}} = \begin{bmatrix} 0 & a_1 & b_1 \\ c_1 & 0 & d_1 \end{bmatrix}$ and $\vec{\mathbf{v}} = \begin{bmatrix} 0 & a_2 & b_2 \\ c_2 & 0 & d_2 \end{bmatrix}$ be vectors in $\mathbb{W}$ and let p, q be real numbers. Then

$$p\vec{\mathbf{u}} + q\vec{\mathbf{v}} = p\begin{bmatrix} 0 & a_1 & b_1 \\ c_1 & 0 & d_1 \end{bmatrix} + q\begin{bmatrix} 0 & a_2 & b_2 \\ c_2 & 0 & d_2 \end{bmatrix} = \begin{bmatrix} 0 & pa_1 & pb_1 \\ pc_1 & 0 & pd_1 \end{bmatrix} + \begin{bmatrix} 0 & qa_2 & qb_2 \\ qc_2 & 0 & qd_2 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & pa_1 + qa_2 & pb_1 + qb_2 \\ pc_1 + qc_2 & 0 & pd_1 + qd_2 \end{bmatrix} \in \mathbb{W}$$

showing that $\mathbb{W}$ is closed under vector addition and scalar multiplication and is thus a subspace.

(b) span$\{S\}$ is a vector space. To see if S is a basis of this vector space, we need to check that the set is linearly independent. This can be done using the Wronskian.

$$W\left[2, 1-t, t+t^3\right](t) = \begin{vmatrix} 2 & 1-t & t+t^3 \\ 0 & -1 & 1+3t^2 \\ 0 & 0 & 6t \end{vmatrix} = -12t$$

Since this is not equal to zero for at least one t, the vectors in S are linearly independent, thus forming a basis for span$\{S\}$. Because there are 3 linearly independent vectors in the set, the dimension of the vector space is 3.

(c)

$$\begin{vmatrix} 2-\lambda & -12 \\ 1 & -5-\lambda \end{vmatrix} = (2-\lambda)(-5-\lambda) + 12 = \lambda^2 + 3\lambda + 2 = (\lambda+1)(\lambda+2) = 0 \implies \lambda = -1, -2$$

$$\lambda = -1: \quad \left[\begin{array}{cc|c} 3 & -12 & 0 \\ 1 & -4 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & -4 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \vec{\mathbf{v}} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$\lambda = -2: \quad \left[\begin{array}{cc|c} 4 & -12 & 0 \\ 1 & -3 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \vec{\mathbf{v}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

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